MATH 231B PARTIAL DIFFERENTIAL EQUATIONS

Lecture 21. The Fredholm Alternative.

Definition. H is a Hilbert space. An operator $F : H \to H$ is *Fredholm* if its null space is finite dimensional, and its range is closed and has finite codimension. (More generally we can make this definition when F is a mapping between two Banach spaces.)

(iii). In general, for $A : H \to H$ bounded, $\overline{R(A)} = N(A^*)^\perp$. Indeed, suppose $u \in R(A)$ and $v \in N(A^*)$. Then $u = Aw$ for some w and

$$(u, v) \ = \ (Aw, v) \ = \ (w, A^*v) \ = \ 0.$$

Hence $R(A) \subset N(A^*)^\perp$. Now suppose $u \in R(A)^\perp$. Then for every w,

$$0 \ = \ (Aw, u) \ = \ (w, A^*u).$$

Hence $u \in N(A^*)$. Thus $R(A)^\perp \subset N(A^*)$, and so $N(A^*)^\perp \subset \overline{R(A)}$ and we have equality. In our case, Since $R(I - K)$ is closed, we get $R(I - K) = N(I - K^*)^\perp$.

Examples. The operators $(I + K)$ for K compact. The shift operators: for an orthonormal base u_1, u_2, ... let $S_+u_j = u_{j+1}$. This operator has null space $\{0\}$ but image is not H. The operator $S_-u_j = \begin{cases} u_{j-1} & j \geq 2 \\ 0 j = 1 \end{cases}$. This operator has null space generated by e_1, but its image is the whole space.

Fredholm Alternative. *Let $K : H \to H$ be a compact linear operator. Then*

(i). The null space $N(I - K)$ is finite dimensional.
(ii). The range of $R(I - K)$ is closed.
(iii). $R(I - K) = N(I - K^*)^\perp$.
(iv). $N(I - K) = \{0\} \Leftrightarrow R(I - K) = H$.
(v). $\dim N(I - K) = \dim N(I - K^*)$.

Corollary. *Either $(I - K)$ is one-to-one and onto and hence $(I - K)^{-1}$ is bounded by the Open Mapping Theorem, or not in which case $N(I - K)$ is non-trivial but finite dimensional and $(I - K)u = f$ has a solution if and only if $f \in N(I - K^*)^\perp$.*

Proof. We dealt with (i) and (ii) last time.

(iv). Suppose $H_1 = (I - K)H \neq H$, but $N(I - K) = \{0\}$. Set $H_2 = (I - K)H_1$, $H_3 = (I - K)H_2, \ldots$. We claim this sequence is decreasing at each step. Indeed, if j is the first number with $H_{j+1} = H_j$ then there exists a non-zero $u \in H_{j-1} \setminus H_j$. But $(I - K)u \in H_j$ and so there exists $v \in H_j$ with $(I - K)v = (I - K)u$. But then $u - v$ is not zero but $(I - K)(u - v) = 0$. This contradicts $N(I - K) = \{0\}$, so the sequence $H, H_1, H_2, \ldots$ is strictly decreasing. Now choose $u_j \in H_j \cap H_{j+1}^\perp$ with $\|u_j\| = 1$. If $\ell > j$ then $(I - K)u_j \in H_{j+1}$ so $Ku_j - Ku_\ell = (u_j - u_\ell) - (I - K)u_j - (I - K)u_\ell$. But $(I - K)u_j$, $(I - K)u_\ell$ and u_ℓ are all in H_{j+1}, so $\|Ku_j - Ku_\ell\| \geq \operatorname{dist}(u_j, H_{j+1}) = 1$. This contradicts K being compact, so $(I - K)H = H$. To go the other way, we apply this result to $(I-K^*)$ and use (iii). Indeed, if $(I-K)H = H$, then $N(I-K^*) = \{0\}$, so $(I-K^*)H = H$, and $N(I-K^*) = \{0\}$.

(v). We show that

$$\dim N(I-K) \geq \dim R(I-K)^\perp \ \ (= \ \dim N(1-K^*))\,.$$

Indeed, if not then we can define a one-to-one linear map $A : N(I-K) \to R(I-K)^\perp$ and extend A to H by setting it equal to zero on $N(I-K)$. Then A has finite rank so $K+A$ is compact and $I-(K+A)$ is injective. But then by (iv) it is also onto. However, this is a contradiction since A does not map onto $R(I-K)^\perp$. This proves the inequality. To get the reverse inequality, just change the roles of K and K^*.

Definition Suppose $A : H \to H$ is a bounded linear operator. The resolvant set is

$$\rho(A) \ = \ \{\eta \in \mathbb{R} : A-\eta I \text{ is one-to-one and onto}\} \ = \ \{\eta \in \mathbb{R} : (A-\eta I)^{-1} \text{ exists and is bounded}\}.$$

The equivalence of the two definitions is by the open mapping theorem or the closed graph theorem.

The spectrum is

$$\sigma(A) \ = \ \mathbb{R} \setminus \rho(A).$$

The point spectrum is the set of eigenvalues

$$\sigma_p(A) \ = \ \{\lambda : A-\lambda I \text{ has non-trivial null space}\} \ = \ \{\lambda : A-\lambda I \text{ is not one-to-one}\}.$$

We can complexify the Hilbert space and make the same definitions with $\mathbb{R}$ replaced by $\mathbb{C}$.

Theorem. *Assume the dimension of H is infinity and $K : H \to H$ is compact. Then*

(i). $0 \in \sigma(K)$.
(ii). $\sigma(K) - \{0\} \ = \ \sigma_p(K) - \{0\}$.
(iii). $\sigma(K) \setminus \{0\}$ *is either a finite set or a sequence tending to* 0.

Proof. (i) Assume $0 \notin \sigma(K)$. Then $K : H \to H$ is invertible. But then $I = K^{-1}K$ is the composition of a compact operator and a bounded operator and hence compact, which is a contradiction.

(ii) Suppose $\eta \in \sigma(K) \setminus \{0\}$. Then the Fredholm alternative implies that $N(K-\eta I)$ is non-trivial, and hence $\eta \in \sigma_p(K)$.

(iii) Now suppose that (η_k) is a sequence of distinct elements of $\sigma(K) - \{0\}$. We will show that if $\eta_k \to \eta$, then $\eta = 0$. However then $\sigma(K)$ is a set bounded by $\|K\|$ whose only limit point is zero. Hence if it is infinite it forms a sequence tending to zero.

Since $\eta_k \in \sigma_p(K)$, there exists $w_k \neq 0$ with $Kw_k = \eta_k w_k$. The vectors $w_1, w_2, \ldots$ must be linearly independent. (Exercise). Let H_k be the space space spanned by

$\{w_1, \dots, w_k\}$. Then $H_k \subset H_{k+1}$ and this is a strict inclusion. Moreover, $(K - \eta_k I)H_k \subseteq H_{k+1}$. Choose $u_k \in H_k \cap H_{k-1}^{\perp}$ with $\|u_k\| = 1$. Then for $k \neq \ell$,

$$\left\| \frac{Ku_k}{\eta_k} - \frac{Ku_\ell}{\eta_\ell} \right\| \;=\; \left\| \frac{(Ku_k - \eta_k u_k)}{\eta_k} \;-\; \frac{(Ku_\ell - \eta_\ell u_\ell)}{\eta_\ell} \;-\; (u_k - u_\ell) \right\| \;\geq\; 1.$$

Hence Ku_k/η_k does not have a convergent subsequence and so neither does Ku_k. This contradicts the compactness of K.