MATH 231B PARTIAL DIFFERENTIAL EQUATIONS

Lecture 22. Fredholm alternative for the Dirichlet Problem.

Recall:

$$Lu \;=\; \sum_{i,j}(a^{ij}u_{x_i})_{x_j} \;+\; \sum_i b^i u_{x_i} \;+\; cu.$$

$$B[u,v] \;=\; \int_U \left(\sum_{i,j}(a^{ij}u_{x_i})_{x_j} \;+\; \sum_i b^i u_{x_i} \;+\; cu \right) dx$$

We showed that there exists γ such that for every $f \in L^2(U)$,

$$\begin{cases} Lu + \gamma u = f & \text{in U} \\ u = 0 & \text{on } \partial U. \end{cases}$$

has a unique weak solution $u \in H_0^1(U)$. For fixed f, this means that for every $v \in H_0^1(U)$,

$$\tilde{B}[u,v] \;=\; B[u,v] + \gamma \int uv\,dx \;=\; \int_U fv\,dx.$$

Note that $\tilde{B}$ is the bilinear form corresponding to the operator $\tilde{L} = L + \gamma I$. We write

$$u = Kf.$$

Then heuristically $K = \tilde{L}^{-1}$. We have

$$\tilde{L}u = f \text{ weakly } \Leftrightarrow \; u = Kf.$$

Expressed another way we have

$$\tilde{B}[Kf, v] \;=\; \int fv \qquad\qquad \text{for all } v \in H_0^1(U).$$

The operator K is compact, because we established an energy inequality for $c > 0$,

$$c\|Kf\|^2_{H^1(U)} \;\le\; \tilde{B}[Kf, Kf] \;=\; \int (Kf)f\,dx \;\le\; \|Kf\|_{H^1(U)}\|f\|_{L^2(U)},$$

so

$$c\|Kf\|_{H_0^1(U)} \;\le\; \|f\|_{L^2(U)},$$

and since by Kondrakov the inclusion of $H_0^1(U)$ in $L^2(U)$ is compact, we see that K is compact from $L^2(U)$ to $L^2(U)$.

Lemma. *The image of K is dense in $L^2(U)$.*

Proof. If $f \in (KL^2(U))^\perp$, then

$$0 \;=\; \int fKf \;=\; \tilde{B}[Kf, Kf],$$

so $Kf = 0$. However, this implies

$$0 \;=\; \tilde{B}[Kf, f] \;=\; \int f^2,$$

and so $f = 0$.

Now we have for $u \in H_0^1(U)$ and $f \in L^2(U)$,

$Lu = f$ weakly $\;\Leftrightarrow\;$ $\tilde{L}u = f + \gamma u$ weakly $\;\Leftrightarrow\;$ $u = Kf + \gamma Ku$ $\;\Leftrightarrow\;$ $(I - \gamma K)u = Kf$.

Notice that

$$Lu = 0 \text{ weakly } \;\Leftrightarrow\; (I - \gamma K)u = 0,$$

From the Fredholm alternative, we see that either $(I - \gamma K)$ is invertible in which case $Lu = f$ always has a unique weak solution $u \in H_0^1(U)$ for every $f \in L^2(U)$, or not, in which case $Lu = 0$ has non-trivial weak solutions, and the space $N \subset H_0^1(U)$ of such solutions is finite dimensional. In addition we see that $\dim N$ equals $\dim N(I - \gamma K^*)$. Furthermore, $Lu = f$ has a weak solution if and only if $Kf \in N(I - \gamma K^*)$.

Our Problem: Interpret $(I - \gamma K^*)$ and the condition $Kf \in N(I - \gamma K^*)$ in terms of L.

Definition. The adjoint bilinear form

$$B^* : H_0^1 \times H_0^1(U) \;\to\; \mathbb{R}$$

is defined by

$$B^*[v, u] \;:=\; B[u, v]$$

for all $u, v \in H_0^1(U)$. We say that u is a weak solution of the equation

$$(*) \qquad \begin{cases} L^*v = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$$

if

$$B^*[v, u] \;=\; \int fu, \qquad\qquad \text{for all } u \in H_0^1(U).$$

Remark. If $a^{ij}, b^i \in C^1(\bar{U})$ and we set

$$L^*v \;=\; -\sum_{i,j=1}^{n} \left(a^{ij} v_{x_j}\right)_{x_i} \;-\; \sum_{i=1}^{n} b^i v_{x_i} \;+\; \left(c - \sum_{i=1}^{n} b^i_{x_i}\right) v,$$

then $u \in C^2(\bar{U})$ is a weak solution of (*) if and only if it is a classical solution.

Lemma. *If $v \in H_0^1(U)$ and $f \in L^2(U)$, then*

$$\tilde{L}^* v = f \text{ weakly} \quad \Leftrightarrow \quad \tilde{K}^* f = v.$$

Proof. Indeed, using the fact that $KL^2(U)$ is dense in $L^2(U)$ we have

$$\begin{aligned} \tilde{L}^* v = f \text{ weakly} \ &\Leftrightarrow \ \tilde{B}^*[v,u] = \int fu \quad \text{for all } u \in H_0^1(U) \\ &\Leftrightarrow \ \tilde{B}[u,v] = \int uf \quad \text{for all } u \in H_0^1(U) \\ &\Leftrightarrow \ \tilde{B}[Kw,v] = \int (Kw)f \quad \text{for all } w \in L^2(U) \\ &\Leftrightarrow \ \tilde{B}[Kw,v] = \int w(K^* f) \quad \text{for all } w \in L^2(U) \\ &\Leftrightarrow \ \int wv = \int w(K^* f) \quad \text{for all } w \in L^2(U) \\ &\Leftrightarrow \ v = K^* f. \end{aligned}$$