Lecture 26. Boundary Regularity.

$$Lu \;=\; -\sum_{i,j}(a^{ij}u_{x_i})_{x_j}$$

$u \in H^1(U)$ is a weak solution of $Lu = f$ if

$$B[u,v] \;=\; \int_U \sum_{i,j} a^{ij}u_{x_i}v_{x_j}\,dx = \int fv \qquad \text{for all } v \in H_0^1(U).$$

Theorem. *Assume U is a bounded open subset of $\mathbb{R}^n$, $a^{ij} \in C^1(\bar{U})$, $b, c \in L^\infty(U)$, $f \in L^2(U)$ and $u \in H_0^1(U)$, and that u is a weak solution of the elliptic PDE*

$$Lu = f \qquad\qquad \text{in } U.$$

Assume also that ∂U is C^2. Then

$$u \in H^2(U),$$

and

$$\|u\|_{H^2(U)} \;\le\; C\left(\|f\|_{L^2(U)} \;+\; \|u\|_{L^2(U)}\right),$$

where $C = C(U, a^{ij}, b^i c)$.

Main Case: U is the upper half ball $B(0,1) \cap \mathbb{R}^n_+$, $V = B(0,1/2) \cap \mathbb{R}^n_+$. We show that

$$\|Du\|^2_{L^2(V)} \;\le\; C\int_U |f|^2 + |u|^2\,dx.$$

$$\|D^2u\|^2_{L^2(V)} \;\le\; C\int_U f^2 + u^2 + |Du|^2\,dx.$$

We show that for $W \subset\subset U$, if u is a weak solution of $Lu = f$ on U, then

To see this, set $v = u$ in the equation

$$B[u,v] \;=\; \int_U fv\,dx,$$

to get

$$\theta\int_U |Du|^2\,dx \;\le\; \int_U \sum_{i,j} a^{ij}u_{x_i}u_{x_j}\,dx \;=\; \int_U \left(f - \sum_i b^i u_{x_i} - cu\right)u\,dx$$

$$\le\; C\left(\int_U f^2\,dx \;+\; \int_U |uDu|\,dx \;+\; \int_U |u|^2\,dx\right),$$

But by Cauchy's theorem,

$$\int_U |uDu|\,dx \ \le\ \varepsilon \int_U |Du|^2\,dx \ +\ \frac{1}{4\varepsilon}\int_U |u|^2\,dx,$$

so by choosing ε sufficiently small, we get

$$\int_U |Du|^2\,dx \ \le\ C\left(\int_U |f|^2\,dx \ +\ \int_U |u|^2\,dx\right).$$

Difference quotient. The difference quotient idea from the interior regularity case works the same way in this case if we choose a cut off function ζ which equals 1 on $B(0,1/2)$ and is supported on $B(0,1)$. What we succeed in proving is

$$\|D_k^h Du\|^2_{L^2(V)} \ \le\ C\left(\int_U |Du|^2\,dx \ +\ \int_U |f|^2\,dx\right) \ \le\ C'\left(\|f\|^2_{L^2(U)} \ +\ \|u\|_{L^2(U)}\right),$$

provided $k < n$. It remains to prove the case $k = n$.

Remark. We showed that if $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$ and $f \in L^2(U)$ and $u \in H^1(U)$ and $Lu = f$ weakly, then $u \in H^2(U)$ and for any $V \subset\subset U$ we have

$$\|D^2 u\|_{H^2(V)} \ \le\ C\left(\|f\|_{L^2(U)} \ +\ \|u\|_{L^2(U)}\right).$$

However, now Lu is defined in the sense of weak derivatives and we can now use the equation for the weak solution

$$B[u,v] \ =\ \int fv, \qquad \text{for all } v \in C_c^\infty(U),$$

to see that

$$\int (Lu - f)v\,dx \ =\ 0 \qquad \text{for all } v \in C_c^\infty(U),$$

and hence $Lu = f$ almost everywhere.

Now to deal with a bound for $u_{x_n x_n}$, we have

$$-a^{nn}u_{x_n x_n} \ =\ fv \ +\ a^{nn}_{x_n}u_{x_n} \ +\ \sum_{i+j<n}(a^{ij}u_{x_i})_{x_j} \ -\ \sum_i b^i u_{x_i} v \ -\ cv$$

We noted that $a_{nn} > \theta > 0$ on U. Hence

$$\|u_{x_n x_n}\|_{L^2(V)} \ \le\ C\left(\|f\|_{L^2(U)} + \|Du\|_{L^2(U)}\right).$$

Higher Regularity Theorem. *Let m be a non-negative integer and assume*

$$a^{ij}, b^i, c \in C^{m+1}(U), \qquad (i,j = 1, \ldots, n),$$

and

$$f \in H^m(U), \qquad u \in H^1(U),$$

and

$$Lu = f \qquad \text{in } U$$

in the weak sense. Then

$$u \in H^{m+2}_{\text{loc}}(U)$$

and for each $V \subset\subset U$,

$$\|u\|_{H^{m+2}(V)} \;\le\; C\left(\|f\|_{H^m(U)} \;+\; \|u\|_{L^2(U)}\right),$$

where $C = C(m, U, V, a^{ij}, b^i, c)$.

Furthermore, if we assume in addition that U is bounded and

$$\partial U \in C^{m+2},$$

and

$$a^{ij}, b^i, c \in C^{m+1}(\bar{U}), \qquad (i,j = 1, \ldots, n),$$

and

$$u \in H^1_0(U),$$

then

$$u \in H^{m+2}(U),$$

and

$$\|u\|_{H^{m+2}(U)} \;\le\; C\left(\|f\|_{H^m(U)} \;+\; \|u\|_{L^2(U)}\right),$$

where $C = C(m, U, a^{ij}, b^i, c)$.

The proof follows the book.