

Lecture 3: Surface geometry.

Last time we defined an n -dimensional smooth manifold M , the tangent space $T_p M$ which is spanned by

$$\partial_{x_1}|_p, \dots, \partial_{x_n}|_p,$$

when $x = (x_1, \dots, x_n)$ are coordinates around p . A metric on M is an inner product $g(p)(\cdot, \cdot) = \langle \cdot, \cdot \rangle_p$ assigned to each $T_p M$ which varies smoothly. In coordinates x we write

$$g_{ij}(p) := \langle \partial_{x_i}, \partial_{x_j} \rangle_p.$$

Remark. The standard metric on \mathbb{R}^n is denoted by $dx_1^2 + \dots + dx_n^2$. Nowadays this is shorthand for $dx_1 \otimes dx_1 + \dots + dx_n \otimes dx_n$. By definition

$$dx_i \otimes dx_j(\partial_{x_k}, \partial_{x_\ell}) = \delta_{ik} \delta_{j\ell}.$$

Definition. 1. If $\phi : M \rightarrow N$ is a smooth map between smooth manifolds, and if $v \in T_p M$, then the push forward $d\phi(v) = \phi_* v \in T_{\phi(p)} N$ is defined by

$$\phi_* v|_{\phi(p)}(f) = v_p(f \circ \phi).$$

If g is a Riemannian metric on N , and ϕ is a map such that $\phi_*(p)$ is injective for every $p \in M$ (i.e. ϕ is an immersion), then ϕ pulls g back to a Riemannian metric $\phi^* g$ on M by the formula

$$\phi^* g(p)(u, v) = g(p)(\phi_* u, \phi_* v).$$

Example. If M is a surface and $\phi : M \rightarrow \mathbb{R}^3$ is an immersion, then $\phi^*(dx^2 + dy^2 + dz^2)$ is the metric which M inherits from \mathbb{R}^3 as an immersed submanifold.

The word metric has two meanings. A Riemannian metric makes a manifold itself into a metric space. Indeed, we have the **Riemannian distance function**: For points p, q on M ,

$$d(p, q) = \inf\{\text{length of } \gamma : \gamma \text{ is a curve joining } p \text{ to } q\}.$$

When we talk about the metric on M we mean the Riemannian metric, that is the bilinear form assigned to each tangent space, unless otherwise stated.

Normal coordinates. If M is a Riemannian manifold with metric g and $p \in M$, then there exist coordinates x in a neighborhood of p such that $x = 0$ at the point p , and written in coordinates,

$$g_{ij}(x) = \delta_{ij} + O(x_1^2 + \dots + x_n^2).$$

These are called **normal coordinates** at p . They are closer to Euclidean coordinates than general coordinates. You usually cannot get rid of the quadratic error to get coordinates which are any closer to being Euclidean.

Definition. If g is a metric on M , and u is a smooth function on M , we say that the metric $e^{2u}g$ is **conformal to** g . Geometrically, the meaning of two metrics being conformal is that the angle between curves is the same when measured in the two metrics - conformal metrics only change lengths, not angles. A map $\phi : M \rightarrow N$ between Riemannian manifolds (M, g) and (N, h) is a **conformal transformation** if g and ϕ^*h are conformal metrics. In two dimensions conformal transformations are very important. Indeed conformal transformations of the plane are either holomorphic or antiholomorphic, and by the Riemann mapping theorem every two simply connected plane domains (neither of which is the whole plane) are conformal. In higher dimensions, the situation is quite different. A conformal transformation of any open subset of \mathbb{R}^n to any other extends to a conformal transformation of $\mathbb{R}^n \cup \infty = S^n$, and can only be a Möbius transformation.

Today we define the volume form dV (or dA for surfaces) for g , and the gradient, divergence and Laplacian, Δ , for g . For the case of surfaces we will define the Gaussian curvature, K . If g and $e^{2u}g$ are conformally related metrics, then

$$K_{e^{2u}g} = e^{-2u} (-\Delta u + K_g).$$

We will not prove this, but next week we will investigate how to find u to make $K_{e^{2u}g}$ constant.

Volume.

Lemma. . If v_1, \dots, v_n are vectors in \mathbb{R}^n , then the volume of the parallelepiped spanned by v_1, \dots, v_n is $\sqrt{\det g_{ij}}$ where $g_{ij} = v_i \cdot v_j$.

Proof. The volume of the parallelepiped is

$$|\det v_{ij}|,$$

where $V = v_{ij}$ is the matrix whose i th row is v_i written in standard cartesian coordinates. But

$$(\det V)^2 = \det V \det V = \det V^* \det V = \det VV^*.$$

However, VV^* is the matrix (g_{ij}) . Indeed,

$$(VV^*)_{ij} = \sum_k v_{ik}v_{jk} = v_i \cdot v_j = g_{ij}.$$

The point here is that we have expressed the volume of the parallelepiped using only the inner products of the edges and not any coordinate system. It justifies the definition of the volume element for the metric as

$$dV = \sqrt{\det g_{ij}} dx_1 \dots dx_n,$$

where g_{ij} is the matrix for g in x coordinates. This volume form can be checked to be independent of the choice of coordinates.

We remark that

$$dV_{e^{2u}g} = e^{nu} dV_g.$$

For surfaces this becomes

$$dA_{e^{2u}g} = e^{2u} dA_g.$$

Curvature. A piece of plane in \mathbb{R}^3 can be bent into a piece of cylinder without changing the lengths of curves within it. Indeed, the cylinder is locally isometric to the plane. However, the plane cannot be changed into a sphere without changing the lengths of curves. The *curvature* of a manifold at a point measures how intrinsically curved the space is at that point. For a surface it is given by a single number $K(p)$, the Gauss curvature. It has several geometric interpretations. For example if $B(p, \varepsilon)$ denotes the ball of points on M which can be joined to p by a curve of length less than ε , then

$$\text{volume}(B(p, \varepsilon)) = \pi \left(\varepsilon^2 - \frac{K(p)}{12} \varepsilon^4 + O(\varepsilon^5) \right).$$

If balls of small radius have smaller volume than balls of the same radius in the plane then the curvature is greater than or equal to zero, and if small balls have larger volume then the curvature is negative. The formula for curvature in terms of the metric g_{ij} is lengthy.

Extrinsic interpretation for surfaces in \mathbb{R}^3 . For a point $p \in M$ where M is a surface embedded in \mathbb{R}^3 , the curvature is the product of the two principle curvatures. While the principal curvatures themselves are extrinsic, their product is intrinsic. This is an observation of Gauss.