Lecture 9. Sobolev Spaces.

Last time: In dimension 1, if v is the weak derivative of u with $u, v \in L^1_{\mathrm{loc}}(\mathbb{R}^1)$ then

$$u(x) \;=\; C \;+\; \int_0^x v(y)\,dy.$$

Remark. In higher dimensions, functions with weak first order partial derivative need not be continuous. (We will see later that on $\mathbb{R}^n$ it takes weak partial derivatives up to order n to guarantee continuity.) The uniqueness of weak derivatives does however imply the following:

Lemma. *Suppose that $u, v \in L^1_{\mathrm{loc}}(U)$ and $Du = v$. If $V \subset U$ is an open set and if $u|_V$ is in $C^1(V)$, then v agrees with the classical derivative of u on V.*

From this we can deduce for example that the characteristic function of the ball $B(0,1)$ has no weak derivative v, for if it did then v would have to equal zero off the set $\partial B(0,1)$, but this is a set of measure zero, and so $v = 0$.
Exercise. If $u \in L^1_{\mathrm{loc}}(U)$ has weak derivatives $D_{x_i}u = 0$ for $i = 1, \ldots, n$, u is constant.

This shows that if $\chi_{B(0,1)}$ had a weak derivative then it would be constant which is clearly a contradiction.

Example 1. If $s < n-1$ then $u(x) = |x|^{-s}$ has first order derivatives in the weak sense.

$$u_{x_i} \;=\; -s x_i |x|^{-s-2}.$$

Proof.

$$\int_{\mathbb{R}^n} u\phi_{x_i}\,dx \;=\; \lim_{\varepsilon\to 0}\int_{U-B(0,\varepsilon)} u\phi_{x_i}\,dx \;=\; -\lim_{\varepsilon\to 0}\left(\int_{U-B(0,\varepsilon)} u_{x_i}\phi\,dx \;+\; \int_{\partial B(0,\varepsilon)} u\phi\nu_i\,dS\right),$$

where $\nu(x) = -x$ is the unit normal on $\partial B(0,1)$. But

$$\left|\int_{\partial B(0,\varepsilon)} u\phi\nu_i\,dS\right| \;\le\; \|\phi\|_\infty \varepsilon^{-s}\int_{\partial B(0,1)} |\nu^i|\,dS \;\le\; \|\phi\|_\infty \varepsilon^{n-1-s}|\partial B(0,1)|.$$

This tends to zero as $\varepsilon \to 0$ provided $n-1-s > 0$. Moreover, under this same condition $u_{x_i} \in L^1_{\mathrm{loc}}$. Indeed, $|u_{x_i}| \le\; C|x|^{-s-1}$ which is locally integrable when $s+1 < n$.

Example 2. Let $\{r_k\}$ be a countable dense subset of $U = B(0,1)$. If $s < n-1$ then

$$u(x) \;=\; \sum_{k=1}^{\infty}\frac{1}{2^k}|x - r_k|^{-s}$$

converges in $L^1(U)$ and has a weak derivative. However it is unbounded on each open subset of U.

Theorem. *(a). If $u \in L^1_{\mathrm{loc}}(U)$ has derivatives $D^\alpha u$ and $D^{\alpha+\beta}u$ in the weak sense, then $D^\alpha(D^\beta u)$ exists and equals $D^{\alpha+\beta}u$.*

(b). If $D^\beta u$ exists in the weak sense for $\beta \leq \alpha$, then for $\zeta \in C_c^\infty(U)$ we have the Leibnitz formula

$$D^\alpha u \;=\; \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u.$$

Proof. See 5.2.3.

Definition. We define the Sobolev space $W^{k,p}(U)$ to be those functions $u \in L^p(U)$ such that for $|\alpha| \leq k$, the derivative $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Example. Note that $|x|^{-s}$ and $D_{x_i}|x|^{-s}$ are in $L^p(B(0,1))$ if $p(s+1) < n$, so $|x|^{-s} \in W^{1,p}(B(0,1))$ if $s < (n-p)/p$. Similarly for the function in Example 2.

Remarks. 1. $H^k(U) := W^{k,2}(U)$ is a Hilbert space.

2. We identify functions in $W^{k,p}(U)$ which agree almost everywhere.

Definition. The norm on the space $W^{k,p}(U)$ for $1 \leq p < \infty$ is

$$\|f\|_{W^{k,p}} \;=\; \left(\sum_{|\alpha|\leq k} \int_U |D^\alpha f(y)|^p \, dy \right)^{1/p}.$$

The norm on $W^{k,\infty}(U)$ is

$$\sum_{|\alpha|\leq k} \operatorname*{ess\,sup}_U |D^\alpha u|.$$

Remark. Sometimes other authors use equivalent norms which are not precisely equal such as

$$\sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^p(\mathbb{R}^n)}.$$

The normed vector space $(V, \|\ \|)$ is a *Banach space* if the corresponding metric space is complete, that is if every Cauchy sequence $v_j \in V$ has a limit.

Theorem. *The space $W^{k,p}(U)$ is a Banach space. (A normed vector space which is complete.)*

Proof. The proof of the triangle inequality follows from the triangle inequality in $L^p(\Omega)$ and ℓ^p.

Completeness: We suppose that $u_j \in W^{k,p}(U)$ is Cauchy. From the completeness of $L^p(U)$, we can choose a subsequence with u_j converging to some u in $L^p(U)$, such that for $0 < |\alpha| \leq k$, $D^\alpha u_j$ converges to some u_α in $L^p(\Omega)$. But then for $\phi \in C_c^\infty(U)$ we have

$$\int_U u_\alpha \phi \, dx \;=\; \lim_{j\to\infty} \int_U D^\alpha u_j \phi \, dx \;=\; \lim_{j\to\infty} (-1)^{|\alpha|} \int_U u_j D^\alpha \phi \, dx \;=\; (-1)^{|\alpha|} \int_U u D^\alpha \phi \, dx.$$

Hence $u_\alpha = D^\alpha u$, and u is the limit of u_j in $W^{k,p}(U)$.