Math 231C Partial Differential Equations

Lecture 1: Second order parabolic equations.

Let U be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary. Last quarter we studied the problem

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{1}$$

Here L is the operator

$$Lu \;=\; -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} \;+\; \sum_{i=1}^{n} b^i u_{x_i} \;+\; cu. \tag{2}$$

Here, $a^{ij}, b^i, c \in L^\infty(U)$ and we have the ellipticity condition

$$\sum_{i,j} a^{ij} \xi_i \xi_j \;\geq\; \theta |\xi|^2. \tag{3}$$

We defined the bilinear form associated to L for $u, v \in H_0^1(U)$,

$$B[u,v] \;=\; \int_U \left(\sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} \;+\; \sum_{i=1}^{n} b^i u_{x_i} v \;+\; cuv \right) dx. \tag{4}$$

Then u is a weak solution of (1) if

$$B[u,v] \;=\; \int_U f v \, dx \tag{5}$$

for all $v \in H_0^1(U)$. We have the energy estimate for B:

$$B[u,u] \;\geq\; \beta \|u\|^2_{H_0^1(U)} \;-\; \gamma \|u\|^2_{L^2(U)}. \tag{6}$$

Write $U_T = U \times [0,T)$. Now we want to solve the equation

$$\begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t = 0\}. \end{cases} \tag{*}$$

Here, L is the operator on U given in (2), but the coefficients a^{ij}, b^i and c are now functions of (x,t). We assume K is uniformly elliptic, that is the estimate (3) holds uniformly for $(x,t) \in U_T$ and $\xi \in \mathbb{R}^n$. The operator $\partial_t \,+\, L$ is said to be *uniformly parabolic* on U_T.

Galerkin's Method. Our approach is to set $u(t)(x) = u(x,t)$, and thus think of u as a function

$$u : [0,T] \to H_0^1(U).$$

The advantage of this is that under this assumption we want to solve

$$u' \;=\; f - Lu \;\in\; (H_0^1(U))^* \;:=\; H^{-1}(U).$$

The best way to think of $u'(x,t)$ (which for fixed t is a distribution in x) is as a map

$$u' : [0,T] \;\to\; H^{-1}(U).$$

The idea of Galerkin's method is to take an increasing sequence of finite dimensional subspaces of $H_0^1(U)$, that is $W_1 < W_2 < \ldots$ and project $H_0^1(U)$ onto W_m with the orthogonal projection P_m. We then find a solution u_m to the projected equation

$$(**) \qquad \begin{cases} u_m' + (P_m L)u_m = P_m f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = P_m g & \text{on } U \times \{t = 0\}. \end{cases}$$

Indeed, by the usual existence theorem for ODEs, we get a unique solution $u_m \in V_m$ to (**) which is in $C^1([0,T], V_m)$. However, we then obtain energy estimates to show that we can take a subsequence of u_m which converges to a *weak* solution of (*). To carry this out we will need the following preliminaries.

1. The space W_m is constructed by choosing an orthonormal base $w_1, w_2, \ldots$ for $L^2(U)$ which is also orthogonal in $H_0^1(U)$, and taking W_m to be the span of $w_1, \ldots, w_m$. We must show that such a basis w_j exsits.
2. The energy estimates give convergence in certain Hilbert spaces. We need to define the spaces $L^2([0,T], H_0^1(U))$ and $L^2([0,T], H^{-1}(U))$ which we will work with.
3. We need to define the notion of a weak solution to (**).

Lemma 1. *There exists an orthonormal base $w_1, w_2, \ldots$ for $L^2(U)$ which is orthogonal in $H_0^1(U)$.*

Proof. Recall that

$$\|u\|^2_{H_0^1(U)} \;=\; \int_U \sum_{i=1}^n u_{x_i}^2 \, dx \;+\; \int_U u^2 \, dx.$$

By the Sobolev inequalities, there exists $\gamma > 0$ such that

$$\int_U \sum_i u_{x_i}^2 \, dx \;\geq\; \gamma \|u\|^2_{H_0^1(U)}, \qquad \text{for all } u \in H_0^1(U).$$

We consider the operator Δ on U. The left hand side is just $B[u,u]$ where B is the bilinear form associated to $-\Delta$. By the Riesz Representation theorem, there is a bounded operator $K : L^2(U) \to H_0^1(U)$ such that u is a weak solution to

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

if and only if $Kf = u$. As an operator from $L^2(U)$ to $L^2(U)$, K is compact. Moreover, it is symmetric and there exists an orthonormal basis of $L^2(U)$ consisting of eigenfunctions of K, let $w_1, w_2, \ldots$ be these orthonormal eigenfunctions. Then $Kw_j = \lambda_j w_j$, so for all $v \in H_0^1(U)$

$$\lambda_j \sum_{i=1}^{n} \int (w_j)_{x_i} v_{x_i}\, dx \;=\; \int_U w_j v\, dx.$$

Setting $v = w_k$ and noting that $\lambda_j, \lambda_k \neq 0$, we see that if $j \neq k$, then w_j and w_k are orthogonal in $H_0^1(U)$.

Definition. Let X be a Banach space. For a function $u : [0,T] \to X$ we say that u is (strongly) measurable if it is the almost everywhere limit of sequence of *simple functions*, where a *simple function* is a function of the form

$$s(t) \;=\; \sum_{j=1}^{n} \chi_{E_j}(t) x_j,$$

where $x_j \in X$ and $\chi_{E_j}(t)$ is the characteristic function of the Lebesgue measurable set $E_j \subset [0,T]$.

If u is strongly measurable and

$$\int_0^T \|u(t)\|\, dt \;<\; \infty$$

then we can define the integral

$$\int_0^T u(t)\, dt \;\in\; X$$

in such a way that

$$\left\| \int_0^T u(t)\, dt \right\| \;\leq\; \int_0^T \|u\|\, dt$$

and for every element $\ell \in X^*$,

$$\ell\left(\int_0^T u(t)\, dt \right) \;=\; \int_0^T \ell(u(t))\, dt.$$

Remark. We say that u is weakly measurable if the function $t \to \ell(u(t))$ is measurable for every $\ell \in X^*$. Strongly measurable implies weakly measurable.