MATH 231C PARTIAL DIFFERENTIAL EQUATIONS

Lecture 10: Higher Regularity. .

$$(*) \qquad \begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t = 0\}, \end{cases}$$

where

$$Lu \;=\; -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} \;+\; \sum_{i=1}^{n} b^i u_{x_i} \;+\; cu.$$

We assume U is open and bounded with smooth boundary, and $a^{ij},\ b^i,\ c\ \in C^\infty(\bar{U})$ are independent of t.

Theorem. *(Higher Regularity). Suppose*

$$\frac{d^k f}{dt^k} \in L^2([0,T], H^{2m-2k}(U)),\ (k = 0, \ldots, m), \qquad g \in H^{2m+1}(U).$$

Suppose in addition that the following compatibility conditions are satisfied:

$$g^0,\ g^1,\ \ldots, g^m \ \in\ H_0^1(U),$$

where g^k is defined recursively by $g_0 = g$ and

$$g^{k+1} \;=\; \partial_t^k f(x,0) \;-\; Lg_k.$$

Then the weak solution u of (*) satisfies

$$\frac{d^k u}{dt^k} \in L^2([0,T], H^{2m+2-2k}(U)), \qquad (k = 0, \ldots, m+1)$$

with bounds.

Interpolation Lemma. *(Theorem 4, 5.9.2.) Assume U is open and bounded with smooth boundary and ∂U is smooth. Suppose m is a non-negative integer. If $u \in L^2([0,T], H^{m+2}(U))$ and $u' \in L^2([0,T], H^m(U))$, then $u \in C([0,T], H^{m+1}(U))$ (after changing u on a set of measure zero if necessary) with*

$$\sup_{t\in[0,T]} \|u(t)\|_{H^{m+1}(U)} \;\le\; C(\|u\|_{H^{m+2}(U)} \;+\; \|u'\|_{H^m(U)}).$$

Proof of the Lemma. We start with the case $m = 0$. Suppose then that $u \in L^2([0,T], H^2(U))$ and $u' \in L^2([0,T], L^2(U))$. Choose a bounded open set V with $U \subset\subset V$, and a bounded linear extension operator $E : H^2(U) \to H^2(V)$ such that for $w \in H^2(U)$, the extension Ew is compactly supported in V. In fact we can assume that E extends to a bounded extension from $L^2(U)$ to $L^2(V)$. Define $\bar{u}(t) = E(u(t))$. Then

$$\|\bar{u}\|^2_{L^2([0,T],H^2(V))} \;=\; \int_0^T \|\bar{u}(t)\|^2_{H^2(V)}\,dt \;\le\; C\int_0^T \|u(t)\|^2_{H^2(V)}\,dt \;=\; \|u\|_{L^2([0,T],H^2(U)}.$$

We claim that u' is bounded in $L^2(U)$. To see this we use difference quotients. Defining

$$D_t^h u(t) \;=\; \frac{u(t+h)-u(t)}{h},$$

we see that

$$\begin{aligned} \|D_t^h\bar{u}\|_{L^2([0,T-h],L^2(V))} \;&\le\; C\|D_t^h\bar{u}\|_{L^2([0,T-h],L^2(U))} \\ &=\; C\left\|\int_0^1 u(t+sh)\right\|_{L^2([0,T-h],L^2(U)} \;\le\; C\|u'\|_{L^2([0,T],L^2(U)}. \end{aligned}$$

But then we mimic the Theorem on difference quotients for scalar valued functions to get

$$\|\bar{u}'\|_{L^2([0,T],L^2(V))} \;\le\; C\|u'\|_{L^2([0,T],L^2(U))}.$$

Indeed, we can take a sequence $h_j \to 0$ with $D_t^h\bar{u} \to w$ weakly in $L^2([0,T],L^2(V))$. We have the required bound on w, so we just need to check that $u' = w$. Choose $\phi \in C_c^\infty([0,T]$ and $v \in L^2(V)$. Then

$$\int_0^T (D_t^h u(t), v)\phi(t)\,dt \;=\; -\int_0^T (u(t),v)D_t^{-h}\phi(t)\,dt.$$

However, as $h = h_j \to 0$, the left hand side converges to

$$\int_0^T (w(t),v)\phi(t)\,dt$$

and the right side converges to

$$-\int_0^T (u(t),v)\phi'(t)\,dt.$$

Hence $u' = w$

Now we assume that $\bar{u} : [0,T] \to H^2(V)$ is smooth as a function of t. Then

$$\left|\frac{d}{dt}\|D\bar{u}(t)\|^2_{L^2(V)}\right| \;=\; |(D\bar{u}(t), D\bar{u}'(t))| \;=\; |(-\Delta\bar{u}\bar{u}(t), \bar{u}(t))| \;\le\; C(\|\bar{u}(t)\|^2_{H^2(V)} + \|\bar{u}'(t)\|^2_{L^2(V)}).$$

Hence

$$\|D\bar{u}(t)\|_{L^2(V)} - \|D\bar{u}(s)\|_{L^2(V)} \;\le\; C(\|\bar{u}(t)\|^2_{L^2([0,T],H^2(V))} \;+\; \|\bar{u}'(t)\|^2_{L^2([0,T],L^2(V))}).$$

However, there exists a value of s with

$$\|D\bar{u}(s)\|_{L^2(V)} \;\le\; \left(\frac{1}{T}\int_0^T \|D\bar{u}(t)\|^2_{L^2(T)}\,dt\right)^{1/2},$$

so we get a uniform bound on $\|D\bar{u}(t)\|_{L^2(V)}$. Hence

$$\|\bar{u}\|_{H^1(V)} \;\le\; C(\|\bar{u}(t)\|^2_{L^2([0,T],H^2(V))} \;+\; \|\bar{u}'(t)\|^2_{L^2([0,T],L^2(V))}).$$

In the general case we get this bound by first extending u to the slightly larger time integral $[-\varepsilon, T+\varepsilon]$ by reflection, and then by convolving with an approximate identity.

The result for higher values of m is proved easily by induction.

Remark. Now in the higher regularity theorem, we can check that the compatibility data $g_0, \ldots, g_m$ are well defined. Indeed, since

$$\frac{d^k f}{dt^k} \;\in\; H^{2m-2k}(U),$$

we have

$$\partial_t^k f(0) \in H^{2m-2k-1}(U),$$

and so by induction,

$$g_k \;\in\; H^{2m-2k+1}(U), \qquad k = 0, \ldots, m.$$