Math 231C Partial Differential Equations

Lecture 11: Higher Regularity. .

$$(*) \qquad \begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t=0\}, \end{cases}$$

where

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu.$$

We assume U is open and bounded with smooth boundary, and $a^{ij},\ b^i,\ c \in C^\infty(\bar{U})$ are independent of t. L is elliptic.

Regularity Assuptions. $g \in H^{2m+1}(U)$.

$$\begin{aligned} f &\in L^2([0,T], H^{2m}(U)) \\ \partial_t f &\in L^2([0,T], H^{2m-2}(U)) \\ &\vdots \\ \partial_t^m f &\in L^2([0,T], L^2(U)) \end{aligned}$$

Define

$$\begin{cases} g_0 = g \\ g_k = \partial_t^{k-1} f(0) - Lg_{k-1} \quad k = 1, 2, \dots, m. \end{cases}$$

By last lecture we have

$$\begin{aligned} g_0 &\in H^{2m+1}(U) \\ g_1 &\in H^{2m-1}(U) \\ g_m &\in H^1(U). \end{aligned}$$

We say that these are *order m regularity conditions on the data (f,g) of (*).* We call the functions $(g_0, \dots, g_m)$ the order m compatibility data, and we say that it satisfies the compatibility conditions if $g_0, dots, g_m \in H_0^1(U)$.

Theorem. . *If u is the weak solution of (*) where the data (f,g) satisfies the order m regularity and compatibility conditions, then the weak solution u of (*) satisfies*

$$\begin{aligned} u &\in L^2([0,T], H^{2m+2}(U)) \\ \partial_t u &\in L^2([0,T], H^{2m}(U)) \\ &\vdots \\ \partial_t^{m+1} u &\in L^2([0,T], L^2(U)). \end{aligned}$$

Proof. We already showed the case $m = 0$. We hence assume that the result is proved for $m \geq$ and prove it for $m + 1$. But we showed that if $g \in H^2(U) \cap H_0^1(U)$ and

$$\begin{aligned} f &\in L^2([0,T], L^2(U)) \\ \partial_t f &\in L^2([0,T], L^2(U)) \end{aligned}$$

(this is weaker than the $m + 1$ conditions) then

$$\begin{aligned} u &\in L^\infty([0,T], H^2(U)) \\ \partial_t u &\in L^2([0,T], H_0^1(U)) \\ \partial_t^2 u &\in L^2([0,T], H^{-1}(U)) \end{aligned}$$

and $\tilde{u} = u'$ is the weak solution to the equation

$$(**) \qquad \begin{cases} \tilde{u}_t - L\tilde{u} = f_t & \text{in } U \times (0,T] \\ \tilde{u} = 0 & \text{on } \partial U \times (0,T] \\ \tilde{u} = g_1 \text{ on } U \times \{t = 0\}. \end{cases}$$

Now assuming that the the data (f, g) satisfies the order m regularity and compatibility conditions, then the data (f_t, g_1) satisfies the order m regularity conditions, with norms bounded by the order $m + 1$ norms for (f, g). In particular, since $f \in L^2([0,T], H^{2m+2}(U))$ and $f_t \in L^2([0,T], H^{2m}(U))$, but the interpolation lemma we have $f_t \in C([0,T], H^{2m+1}(U))$ with bounds. Since we also have $g \in H^{2m+3}(U)$ we get

$$g_1 \;=\; f_t(0) - Lg_0 \;\in\; H^{2m+1}(U).$$

The compatibility data for this new problem is $(g_1, \ldots, g_m)$ which we already know satisfies the compatibility conditions. We conclude that $\tilde{u}$ satisfies the order m conclusions with bounds in terms of the original data. Hence

$$\begin{aligned} \partial_t u &\in L^2([0,T], H^{2m+2}(U) \\ &\vdots \\ \partial_t^{m+1} u &\in L^2([0,T], L^2(U)) \end{aligned}$$

with bounds in terms of the original data. But for almost every t we have

$$Lu \;=\; f - u_t.$$

By elliptic regularity, there exists $C = C(L, U)$ so that for such t we have

$$\begin{aligned} \|u(t)\|^2_{H^{2m+4}(U)} &\leq C\left(\|f(t) - u_t(t)\|^2_{H^{2m+2}(U)} + \|u(t)\|^2_{L^2(U)}\right) \\ &\leq 2C\left(\|f(t)\|^2_{H^{2m+2}(U)} + \|\tilde{u}\|^2_{H^{2m+2}(U)} + \|u(t)\|^2_{L^2(U)}\right). \end{aligned}$$

Integrating this up in t gives $\partial_t^{m+2} u \in L^2([0,T], H^{2m+4}(U))$ with bounds in terms of the original data.

History. We pause to read the information on Sobolev and Galerkin from Wikipedia http://en.wikipedia.org

Theorem. *(Characterization of H^k by Fourier transform.) Let $k \geq 0$. Then a function $f \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if*

$$(1 + |y|^k)\hat{f} \;\in\; L^2(\mathbb{R}^n).$$

There exists $C > 0$ such that

$$\frac{1}{C}\|u\|_{H^k(\mathbb{R}^n)} \;\leq\; \|(1 + |y|^k)\hat{f}\|_{L^2(\mathbb{R}^n)} \;\leq\; C\|u\|_{H^k(\mathbb{R}^n)}.$$

Moreover, $H^{-1}(\mathbb{R}^n)$ is the set of distributions in $\mathcal{S}'$ (or if you prefer, the dual space of $H^1(\mathbb{R}^n)$) such that $(1 + |y|)^{-1}\hat{u}$ is in $L^2(\mathbb{R}^n)$. (As above, we have an equivalence of norms.)

Maximum Principles.

$$Lu \;=\; -\sum_{i,j=1}^{n} a^{ij}u_{x_i x_j} \;+\; \sum_{i=1}^{n} b_i u_{x_i} \;+\; cu.$$

a^{ij}, b^i, c continuous (may depend on t). L uniformly parabolic.

The *parabolic boundary* of $U_T = U \times (0, T]$ is $\Gamma_T = \bar{U}_T - U_T$.

Weak Maximum Principle. . *Assume $u \in C^{2,1}(U_T) \cap C(\bar{U}_T)$ and $c \equiv 0$. Then*

$$u_t + Lu \leq 0 \text{ in } U \quad \text{(i.e. } u \text{ is a subsolution)} \qquad \Rightarrow \qquad \max_{\bar{U}_T} u = \max_{\Gamma_T} u.$$

Similarly,

$$u_t + Lu \geq 0 \text{ in } U \quad \text{(i.e. } u \text{ is a supersolution)} \qquad \Rightarrow \qquad \min_{\bar{U}_T} u = \min_{\Gamma_T} u.$$

Proof. The second statement follows from the first by considering $-u$. For the first statement, assume $u_t + Lu < 0$ in U_T, but suppose there exists a point $(x_0, t_0) \in U_T$ with $u(x_0, t_0) = \max_{\bar{U}_T} u$. Then u is a maximum on the slice $U \times \{t = t_0\}$ and so $D_{(x,t)}u(x_0, t_0) = 0$ and by ellipticity, $Lu(x_0, t_0) \geq 0$. Since $u_t < -Lu$, we get $u_t < 0$.

In the general case write

$$u^\varepsilon(x, t) \;=\; u(x, t) - \varepsilon t.$$

Then $u_t^\varepsilon - Lu < 0$ and so u^ε attains its max on Γ_T. Hence taking the limit as $\varepsilon \to 0$, so does u. (Exercise.)