

**Lecture 12: Harnack's Inequality.**

**Remark from last time.** If  $u$  is the solution to

$$(*) \quad \begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0, T] \\ u = g & \text{on } U \times \{t = 0\}, \end{cases}$$

and  $f \in C^\infty([0, T] \times \bar{U})$ ,  $g \in C^\infty(\bar{U})$  then  $u \in C^\infty([0, T] \times \bar{U})$ .

There is a version of the maximum principle with  $c \geq 0$ .

Suppose  $U$  is open and bounded in  $\mathbb{R}^n$  with smooth boundary,

$$Lu = - \sum_{i,j} a^{ij} u_{x_i x_j} + \sum_i b^i u_{x_i} + cu,$$

where  $a^{ij} = a^{ji}$  and  $\partial_t + L$  is uniformly parabolic.

**Parabolic Harnack Inequality.** Assume  $u \in C_1^2(U_T)$  solves

$$u_t + Lu = 0 \quad \text{in } U_T,$$

and

$$u \geq 0 \quad \text{in } U_T.$$

Suppose  $V \subset\subset U$  is connected. Then for each  $0 < t_1 < t_2 \leq T$ , there exists a constant  $C$  such that

$$\sup_V u(\cdot, t_1) \leq C \inf_V u(\cdot, t_2).$$

The constant  $C$  depends only on  $V$ ,  $t_1$ ,  $t_2$ , and the coefficients of  $L$ . This is true if the coefficients are bounded (measurable).

In the proof, Evans assumes  $b^i = c = 0$  and  $a^{ij}$  is smooth. We will assume that  $L = -\Delta$  to make the proof simple.

**Proof.** We will show that the inequality holds when  $V$  is an open ball. The general case follows by covering  $V$  with open balls whose closures lie in  $U$ , and dividing the interval  $[t_1, t_2]$  into as many intervals as there are balls, and then iterating the result using points  $x_j$  in the intersection of balls to finally bound  $v(x_1, t_1)$  above by  $v(x_2, t_2)$  plus a constant.

We want to show that there exists  $c > 0$  such that for  $x_1, x_2 \in V$ ,

$$\frac{u(x_1, t_1)}{u(x_2, t_2)} \leq c.$$

Put another way, setting  $v = \log u$  we want a constant  $C$  so that

$$v(x_2, t_2) - v(x_1, t_1) \geq C.$$

Let us see what equation  $v$  satisfies. Indeed, setting  $u = e^v$  in  $u_t - \Delta u = 0$  we get

$$0 = (e^v)_t - \sum_i (e^v)_{x_i x_i} = e^v \left( v_t - \sum_i v_{x_i x_i} - \sum_i v_{x_i} v_{x_i} \right),$$

so

$$v_t = \Delta v + |Dv|^2,$$

which is a non-linear (semi-linear) parabolic equation for  $v$ . Our aim is to show that for  $\mu$  sufficiently large, on  $V \times [t_1, t_2]$  we have

$$(***) \quad \Delta v + \frac{1}{2}|Dv|^2 + \mu t \geq 0.$$

This will prove (\*\*), because setting  $\beta = \mu t_2$  we get

$$(**) \quad v_t \geq \frac{1}{2}|Dv|^2 - \beta.$$

Then

$$\begin{aligned} v(x_2, t_2) - v(x_1, t_1) &= \int_0^1 \frac{d}{ds} v(sx_2 + (1-s)x_1, st_2 + (1-s)t_1) ds \\ &= \int_0^1 Dv \cdot (x_2 - x_1) + v_t(t_2 - t_1) ds \\ &\geq \int_0^1 -|Dv||x_2 - x_1| + (t_2 - t_1)(\alpha|Dv|^2 - \beta) ds \\ &\geq -\gamma. \end{aligned}$$

We set

$$w = \Delta v, \quad \tilde{w} = |Dv|^2, \quad \hat{w} = \Delta v + \frac{1}{2}|Dv|^2.$$

Our aim is to get a lower bound for  $\hat{w}$ . To do this we shall make a maximum principle, and we start by deriving a parabolic differential equation that it satisfies. We have

$$\begin{aligned} w_t &= \Delta v_t = \Delta(\Delta v + |Dv|^2) = \Delta^2 v + \Delta|Dv|^2. \\ \Delta w &= \Delta^2 v \\ 2Dv \cdot Dw &= 2Dv \cdot D\Delta v, \end{aligned}$$

Then

$$\begin{aligned}
w_t - \Delta w - 2Dv \cdot Dw &= \Delta|Dv|^2 - 2Dv \cdot D\Delta v \\
&= \sum_{i,j} \left( 2v_{x_i x_j}^2 + 2v_{x_i} v_{x_i x_j x_j} - 2v_{x_i} v_{x_i x_j x_j} \right) \\
&= 2|D^2 v|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\tilde{w}_t &= 2Dv \cdot Dv_t = 2Dv \cdot D(\Delta v + |Dv|^2) = 2Dv \cdot D\Delta v + 2Dv \cdot D|Dv|^2. \\
\Delta \tilde{w} &= \Delta|Dv|^2 \\
2Dv \cdot D\tilde{w} &= 2Dv \cdot (D|Dv|^2),
\end{aligned}$$

Hence

$$\tilde{w}_t - \Delta \tilde{w} - 2Dv \cdot D\tilde{w} = 2Dv \cdot D\Delta v - \Delta|Dv|^2 = -2|D^2 v|^2.$$

Hence

$$\hat{w}_t - \Delta \hat{w}_t - 2Dv \cdot \hat{w}_t = |D^2 v|^2.$$

This right hand side is positive which is good news since we are trying to bound  $\hat{w}$  below. However, since we know nothing about the boundary values of  $\hat{w}$ , we will need to introduce a cut off function. Choose  $\zeta$  supported in  $U \times (0, T]$  with  $\zeta = 1$  on  $V \times [t_1, t_2]$ . Consider

$$W = \zeta^4 \hat{w} + \mu t,$$

where  $\mu > 0$  will be chosen later, and suppose to get a contradiction that the minimum of  $W$  on  $\bar{U}_T$  is negative. This it is attained at some point  $(x_0, t_0)$ , which must necessarily be in  $U_T$ . At  $(x_0, t_0)$  we have

$$DW = 0,$$

so

$$\zeta^4 D\hat{w} = -(D\zeta^4)\hat{w} = -4\zeta^3(D\zeta)\hat{w}.$$

But at  $(x_0, t_0)$ ,

$$\begin{aligned}
0 &\geq W_t - \Delta W - 2Dv \cdot DW \\
&= \zeta^4(\hat{w}_t - \Delta \hat{w} - 2Dv \cdot D\hat{w}) + ((\zeta^4)_t - \Delta(\zeta^4) - 2Dv \cdot D(\zeta^4)) \hat{w} \\
&\quad - 2D(\zeta^4) \cdot D\hat{w} + \mu \\
&= \zeta^4|D^2 v|^2 + ((\zeta^4)_t - \Delta(\zeta^4) + 8\zeta^2|D\zeta|^2 - 2Dv \cdot D(\zeta^4)) \hat{w} + \mu \\
&\geq \zeta^4|D^2 v|^2 - C(\zeta^2 + |\zeta^3||Dv|)|\hat{w}| + \mu.
\end{aligned}$$

However, since we assume that  $\zeta^2 \hat{w} + \mu t$  is negative at  $(x_0, t_0)$  and  $\mu > 0$ , we see that  $\hat{w} < 0$  at  $(x_0, t_0)$  and so at  $(x_0, t_0)$ ,

$$|Dv|^2 \leq -\Delta v \leq |D^2v|,$$

and

$$|\hat{w}| \leq -\Delta v \leq |D^2v|.$$

and so

$$0 \geq \zeta^4 |D^2v|^2 - C(\zeta^2 + |\zeta^3||Dv|)|D^2v| \geq \zeta^4 |D^2v|^2 - C\zeta^2 |D^2v| - C|\zeta^3||Dv|^{3/2} + mu.$$

But by choosing  $\mu$  sufficiently large, we get a contradiction since the function

$$f(y) = y^4 - Cy - Cy^3$$

is bounded below. Hence for  $\mu$  large, we have  $W = \zeta^4 \hat{w} + \mu t \geq 0$ , and so  $\hat{w} \geq -\mu t_2$  on  $V \times [t_1, t_2]$ .