

Lecture 13: Hyperbolic Equations. .

Remark. Last time we proved the Harnak inequality with the assumption $u > 0$. We can reduce this to $u \geq 0$ by considering $u + \varepsilon$ and taking limits.

Exercise. If U is bounded with smooth boundary and $g \in C(\Gamma_T)$ and u is the weak solution to

$$\begin{cases} u_t + Lu = 0 & \text{in } U_T \\ u = g & \text{on } \Gamma_T, \end{cases}$$

then $u \in C(\bar{U}_T) \cap C^\infty(U_T)$.

First note that the interior regularity is achieved by taking cut off functions. The solution with g non-vanishing on $\partial U \times [0, T]$ is also achieved by taking cut offs. The continuity up to the boundary is achieved by approximating the data by smooth functions and use the weak maximum principle to get convergence of the solutions.

Theorem: Strong Maximum Principle. *If $u \in C^{2,1}(U \times (0, T]) \cap C(\bar{U}_T)$ and $u_t + Lu$ is uniformly parabolic where*

$$L = - \sum_{i,j} a^{ij} u_{x_i x_j} + \sum_i b^i u_{x_i},$$

and a^{ij}, b^i smooth, then

(i). *If $u_t + Lu \leq 0$ in U_T and u attains its maximum over \bar{U}_T at a point $(x_0, t_0) \in U_T$, then u is constant on U_{t_0} .*

(ii). *Similarly if $u_t + Lu \geq 0$ in U_T and u attains its minimum over \bar{U}_T at a point $(x_0, t_0) \in U_T$, then u is constant on U_{t_0} .*

Remark. If

$$\begin{cases} u_t + Lu = 0 & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0, T], \end{cases}$$

and at time $t = 0$, the function u vanishes except on a small ball where $u > 0$, then $u > 0$ on U_T , so there is an infinite speed of propagation of disturbances.

Proof of the SMP. We just prove (i). Assume $u_t + Lu \leq 0$ and u attains its maximum value M at $(x_0, t_0) \in U_T$. Choose connected $W \subset\subset U$ with smooth boundary and $x_0 \in W$ and let Δ_T denote the parabolic boundary of W_T , and let v solve

$$\begin{cases} v_t + Lv = 0 & \text{in } W_T, \\ v = u & \text{on } \Delta. \end{cases}$$

Since $(u-v)_t + L(u-v) \leq 0$ and $u-v \leq 0$ on Δ_T . By the weak maximum principle, $v \leq u$. Now consider $w = M - v$. Then $w_t + Lw = 0$ in W_T and $w \geq 0$. But by the Harnack inequality, for $t \leq t_0$,

$$\max_W w(\cdot, t) \leq \int_W w(\cdot, t_0).$$

Hence $w = 0$ on W_{t_0} . Hence $v = M$ on W_{t_0} . Since $v \leq u \leq M$ we get $u = M$ on W_{t_0} . The same conclusion holds for all connected sets W' which intersect W . Covering U with a countable collection of such sets gives $u = M$ on W_{t_0} .

Hyperbolic Equations..

Suppose $U \subset \mathbb{R}^n$ is open and bounded and write $U_T = U \times (0, T]$. We will study the problem

$$(*) \quad \begin{cases} u_{tt} + Lu = f & \text{in } U_T, \\ u = 0 & \text{on } \partial U \times [0, T] \\ u = g, \quad u_t = h \text{ on } U \times \{t = 0\}. \end{cases}$$

As before,

$$L = - \sum_{i,j} a^{ij}(u_{x_i})_{x_j} + \sum_i b^i u_{x_i} + cu,$$

where a^{ij}, b^i, c may depend on t . We say that $\partial_t^2 + L$ is *uniformly hyperbolic* if there exists $\theta > 0$ such that

$$\sum_{i,j} a^{ij}(x, t) \xi_i \xi_j \geq \theta |\xi|^2, \quad \text{for all } (x, t) \in U_T, \quad \xi \in \mathbb{R}^n.$$

The bilinear form on $H_0^1(U)$ to L at time t is

$$B_t[u, v] = \int_U \left(\sum_{i,j} u_{x_i} v_{x_j} + \sum_i b^i u_{x_i} v + cuv \right) dx.$$

Weak Solution. Suppose

$$\begin{aligned} a^{ij}, b^i, c &\in C^1(\bar{U}_T), \quad a^{ij} = a^{ji}, \\ f &\in L^2(U_T) = L^2([0, T], L^2(U)), \\ g &\in H_0^1(U), \quad h \in L^2(U). \end{aligned}$$

and

$$u \in L^2([0, T], H_0^1(U)), \quad u_t \in L^2([0, T], L^2(U)), \quad u_{tt} \in L^2([0, T], H^2(U)).$$

Then u is a weak solution to (*) if

$$\langle u'', v \rangle + B_t[u, v] = (f, v), \quad \text{for all } v \in H_0^1(U),$$

and

$$u(0) = g, \quad u'(0) = h.$$

Since this implies $u \in C([0, T], L^2(U))$ and $u' \in C([0, T], H^{-1}(U))$, the boundary conditions make sense.

Galerkin Method Let w_1, w_2, \dots be an orthonormal base for $L^2(U)$ which is orthogonal in $H_0^1(U)$. We solve the projected weak equation for

$$u_m = \sum_{k=1}^m c_k^m(t) w_k,$$

which is

$$\begin{aligned} (u_m'', w_k) + B_t[u_m, w_k] &= (f, w_k), & k = 1, 2, \dots, m. \\ (u_m(0), w_k) &= (f, w_k), & (u_m'(0), w_k) = (g, w_k), & k = 1, 2, \dots, m. \end{aligned}$$

Written out in terms of the coefficients this is

$$\begin{aligned} c_m^k''(t) + \sum_{\ell=1}^k c_m^\ell(t) B_t[w_k, w_\ell] &= (f, w_k) \\ c_m^k(0) &= (f, w_k), & c_m^k'(0) &= (g, w_k). \end{aligned}$$

Now $(f, w_k) \in L^2([0, T])$, $B_t[w_k, w_\ell] \in C^1([0, T])$. Hence we get a solution with $c_m^k \in H^2([0, T])$.