Lecture 14: Existence. .

$$(*)\qquad \begin{cases} u_{tt} + Lu = f & \text{in } U_T, \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g,\ u_t = h \ \text{ on } U \times \{t=0\}. \end{cases}$$

Assume

$$\begin{aligned} &a^{ij}, b^i, c \in C^1(\bar{U}_T), \qquad a^{ij} = a^{ji}, \\ &f \in L^2(U_T) = L^2([0,T], L^2(U)), \\ &g \in H_0^1(U),\ h \in L^2(U). \end{aligned}$$

Last time we showed that taking $w_1, w_2, \dots$ an orthonormal base for $L^2(U)$ which is orthogonal in $H_0^1(U)$, we can find

$$u_m(t) \ = \ \sum_{k=1}^{m} c_m^k(t) w_k$$

in $H^2[0,T]\, H_0^1(U)$ such that for all $v \in W_m$,

$$(**)\qquad \begin{aligned} &(u_m(t), v) \ + \ B_t[u_m, v] \ = \ (f, v) \\ &(u_m(0), v) = (g, v), \qquad\qquad (u_m'(0), v) = (h, v). \end{aligned}$$

Energy Estimates. There exists a constant $C = C(U, L, T)$ such that for $m = 1, 2, \dots$ we have

$$\max_{0 \le t \le T} \Big(\|u_m(t)\|_{H_0^1(U)} \ + \ \|u_m'(t)\|_{L^2(U)} \Big) \ + \ \|u_m''\|_{L^2([0,T], H^{-1}(U))}$$
$$\le \ C \Big(\|f\|_{L^2([0,T], L^2(U))} \ + \ \|g\|_{H_0^1(U)} \ + \ \|h\|_{L^2(U)} \Big).$$

Proof. Set $v = u'(t)$ in (**). Then

$$(u_m'', u_m') \ + \ B_t[u_m, u_m'] \ = \ (f, u_m').$$

We can write this as

$$\frac{1}{2}\frac{d}{dt}(u_m', u_m') + \frac{1}{2}\frac{d}{dt}\sum_{i,j} a^{ij} u_{mx_i} u_{mx_j} - \frac{1}{2}\sum_{i,j} a_t^{ij} u_{mx_i} u_{mx_j} + \sum_i b^i u_{mx_i} u' + cuu' \ = \ (f, u_m').$$

Setting

$$A_t[u,v] \;=\; \sum_{i,j} a^{ij} u_{x_i} v_{x_j},$$

we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u_m'\|^2_{L^2(U)} \;+\; A_t[u_m,u_m]\right) \;\le\; C\left(\|u_m\|^2_{H^1_0(U)} \;+\; \|u_m'\|^2_{L^2(U)} \;+\; \|f\|_{L^2(U)}\right)$$
$$\le\; C\left(A_t[u_m,u_m] \;+\; \|u_m'\|^2_{L^2(U)} \;+\; \|f\|^2_{L^2(U)}\right).$$

Applying Gronwall's inequality using the integrating factor e^{-Ct}, we get

$$e^{Ct}\left(\|u_m'(t)\|^2_{L^2(U)} \;+\; A_t[u_m,u_m]\right) \;\le\; \|u_m'(0)\|^2_{L^2(U)} + A_0[u_m(0),u_m(0)] + \|f\|^2_{L^2([0,T],L^2(U))}.$$

Hence

$$\sup_{0\le t\le T}\left(\|u(t)\|^2_{H^1_0(U)} \;+\; \|u'(t)\|^2_{L^2(U)}\right) \;\le\; \left(\|h\|^2_{L^2(U)} \;+\; \|g\|^2_{H^1_0(U)} \;+\; \|f\|^2_{L^2([0,T],L^2(U))}\right).$$

To get the bound on u_m'', use the equation

$$(***) \qquad (u_m'',v) \;=\; (f,v) \;-\; B_t[u_m,v].$$

for $v\in W_m$. However, for general $v\in H^1_0(U)$ with $\|v\|_{H^1_0(U)}\le 1$, write $v=v_1+v_2$ with $v_1\in W_m$ and $v_2\in W_m^\perp$. Then

$$(u'',v) \;=\; (f,v_1) \;-\; B_t[u_m,v_1],$$

so

$$|(u'',v)| \;\le\; C\left(\|f\|_{L^2(U)} \;+\; \|u_m\|_{H^1_0(U)}\right).$$

Hence $\|u''\|_{H^{-1}(U)}$ is bounded by the right hand side. Squaring and integrating we get

$$\|u''\|^2_{L^2([0,T],H^{-1}(U))} \;\le\; C\left(\|f\|^2_{L^2([0,T],L^2(U))} \;+\; \|f_m\|^2_{L^2([0,T],H^1_0(U))}\right).$$

The last term on the right is already bounded in terms of the data f,g,h.

Existence. We take weak limits following 7.2.2c, except we use the following trick to verify that u satisfies the initial conditions:
To show $u(0)=g$ and $u_m'(0)=h$ for $v\in W_m$. Take $v\in W_k$ and consider $\phi(t)v$ with $\phi\in C_c^\infty[0,T)$ and $\phi(0)=1$. Then for $m\ge k$,

$$\int_0^T (u_m'(t),\phi(t)v) \;+\; (u_m(t),\phi'(t)v)\,dT \;=\; -(u_m(0),v(0)) \;=\; -(g,v).$$

However, taking the limit as $m=m_\ell\to\infty$ this gives

$$(***) \qquad \int_0^T \langle u'(t),\phi(t)v\rangle \;+\; (u,\phi'(t)v)\,dt \;=\; (g,v).$$

But the left hand side is $(u(0),v)$. Similarly one shows $u'(0)=h$.