MATH 231C PARTIAL DIFFERENTIAL EQUATIONS

Lecture 15: Uniqueness. .

$$(*) \qquad \begin{cases} u_{tt} + Lu = f & \text{in } U_T, \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g, \ u_t = h \ \text{ on } U \times \{t=0\}. \end{cases}$$

Assume

$$\begin{aligned} &a^{ij}, b^i, c \in C^1(\bar{U}_T), \qquad a^{ij} = a^{ji}, \\ &f \in L^2(U_T) = L^2([0,T], L^2(U)), \\ &g \in H_0^1(U), \ h \in L^2(U). \end{aligned}$$

Uniqueness Theorem. *A weak solution u of (*) with $u \in L^2([0,T], H_0^1(U))$, $u' \in L^2([0,T], L^2(U))$, $u'' \in L^2([0,T], H^{-1}(U))$ is unique.*

Proof. We assume that we have two solutions and take the difference, which solves

$$(*) \qquad \begin{cases} u_{tt} + Lu = 0 & \text{in } U_T, \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = 0, \ u_t = 0 \ \text{ on } U \times \{t=0\}. \end{cases}$$

If only we could say

$$\frac{d}{dt}\frac{1}{2}(u', u') \ = \ 2\langle u'', u\rangle,$$

we could use the technique which we used to get existence to bound $\|u\|_{L^\infty([0,T],H_0^1(U))}$ in terms of the data. Instead we consider

$$w(t) \ = \ \int_0^t u(\tau)\, d\tau, \qquad\qquad v(t) \ = \ w(s) - w(t) \ = \ \int_t^s u(\tau)\, d\tau,$$

where $0 \le s \le T$. Then $w \in C([0,T], H_0^1(U))$ is absolutely continuous with $w' = u$ almost everywhere.

Then

$$\int_0^s \langle u'', v\rangle \ + \ B_t[u, v]\, dt \ = \ 0.$$

We can integrate by parts, and since $u'(0) = 0$ and $v(s) = w(s) - w(s) = 0$, and $v' = -u$, we get

$$\int_0^s (u', u) \ - \ B_t[v', v]\, dt \ = \ 0.$$

Setting

$$A_t[u,v] \;=\; \sum_{i,j} a^{ij} u_{x_i} v_{x_j},$$

We write this as

$$\int_0^s \frac{d}{dt}\frac{1}{2}\left(\|u\|^2_{L^2(U)} \;-\; A_t[v,v]\right)\,dt \;=\; \int_0^s \left(-\sum_{i,j} a^{ij}_t u_{x_i} v_{x_j} \;-\; \sum_i b^i u_{x_i} v \;-\; cuv\right)\,dt$$

Integrating, this becomes

$$\frac{1}{2}\left(\|u(s)\|^2_{L^2(U)} \;+\; A_0[v(0),v(0)]\right) \;\le\; C\int_0^s \left(\|u(t)\|^2_{H^1_0(U)} \;+\; \|v(t)\|^2_{H^1_0(U)}\right)\,dt.$$

Setting $v(t) = w(s) - w(t)$, this becomes

$$\begin{aligned}\|u(s)\|^2_{L^2(U)} \;+\; A_0[w(s),w(s)] \;&\le\; C\int_0^s \left(\|u(t)\|^2_{H^1_0(U)} \;+\; \|w(s)-w(t)\|^2_{H^1_0(U)}\right)\,dt\\ &\le\; Cs\|w(s)\|^2_{H^1_0(U)} \;+\; C\int_0^s \left(\|u(t)\|^2_{H^1_0(U)} \;+\; \|w(t)\|^2_{H^1_0(U)}\right)\,dt.\end{aligned}$$

Hence

$$\|u(s)\|^2_{L^2(U)} \;+\; (\alpha - Cs)\|w(s)\|^2_{H^1_0(U)} \;\le\; C\int_0^s \left(\|u(t)\|^2_{H^1_0(U)} \;+\; \|w(t)\|^2_{H^1_0(U)}\right)\,dt.$$

Choosing T_1 small enough so that $CT_1 < \alpha/2$, we get that for $0 \le s \le T$ we have

$$\|u(s)\|^2_{L^2(U)} \;+\; \|w(s)\|^2_{H^1_0(U)} \;\le\; C_1\int_0^s \left(\|u(t)\|^2_{H^1_0(U)} \;+\; \|w(t)\|^2_{H^1_0(U)}\right)\,dt.$$

Applying Gronwall's inequality we conclude that $u = 0$ on $[0,T_1]$. Applying this similarly to the intervals $[T_1, 2T_1], [2T_1, 3T_1], \ldots$ we conclude $u = 0$ on $[0,T]$.

Regularity We follow 7.2.3.