Lecture 2: Weak Solutions.

Definition. Let X be a Banach space. For a function $u : [0,T] \to X$ we say that u is (strongly) measurable if it is the almost everywhere limit of sequence of *simple functions*, where a *simple function* is a function of the form

$$(*) \qquad s(t) \; = \; \sum_{j=1}^{n} \chi_{E_j}(t) x_j,$$

where $x_j \in X$ and $\chi_{E_j}(t)$ is the characteristic function of the Lebesgue measurable set $E_j \subset [0,T]$.

Equivalently, $u : [0,T] \to X$ is measurable if the inverse image of every open set (and hence any Borel set) is Lebesgue measurable.

For the simple function in (*) we define

$$\int_X s(t)\, dt \; = \; \sum_{j=1}^{n} |E_j| x_j.$$

If u is strongly measurable we say that u is *integrable* if there exists a sequence of simple functions s_j with

$$\int_0^T \|u(t) - s_j(t)\|\, dt \; \to \; 0, \qquad j \to \infty.$$

In this case we set

$$\int_0^T u(t)\, dt \; = \; \lim_{j\to\infty} \int_0^T s_j(t)\, dt.$$

Theorem. *(Bochner) A (strongly) measurable function $u : [0,T] \to X$ is integrable if and only if $t \to \|u(t)\|$ is integrable. We have*

$$\left\| \int_0^T u(t)\, dt \right\| \; \leq \; \int_0^T \|u\|\, dt$$

and for every element $\ell \in X^*$,

$$\ell \left(\int_0^T u(t)\, dt \right) \; = \; \int_0^T \ell(u(t))\, dt.$$

Definition. The space $L^p([0,T], X)$ is the space of measurable functions $u : [0,T] \to X$ such that

$$\|u\|_{L^p([0,T],X)} \; = \; \left(\int_0^T \|u(t)\|^p\, dt \right)^{1/p} \; < \; \infty.$$

Similarly, $C([0,T],X)$ is the space of continuous functions $u : [0,T] \to X$ with

$$\|u\|_{C([0,T],X)} \;=\; \max_{t\in[0,T]} \|u(t)\| \;<\; \infty.$$

Definition. $H^{-1}(U)$ is the dual space to $H_0^1(U)$, that is the space of linear maps $f : H_0^1(U) \to \mathbb{R}$. We write $\langle\ ,\ \rangle$ for the pairing between $H^{-1}(U)$ and $H_0^1(U)$. The norm on $H^{-1}(U)$ is defined by

$$\|f\|_{H_0^1(U)} \;=\; \sup\{\langle f,u\rangle : u \in H_0^1(U),\ \|u\|_{H_0^1(U)} \le 1\}.$$

In particular, if $f_0,\dots,f_n \in L^2(U)$ then we can define $f \in H^{-1}(U)$ by

$$\langle f,u\rangle \;=\; \int_U \left(f_0 u \;+\; \sum_{i=1}^n f_i u_{x_i} \right) dx.$$

Indeed,

$$|\langle f,u\rangle| \;\le\; \|(f_0,f_1,\dots,f_n)\|_{L^2(U)} \|(u,u_{x_1},\dots,u_{x_n})\|_{L^2(U)} \;=\; \left(\int_U \sum_{i=0}^n f_i^2 \, dx \right)^{1/2} \|u\|^2_{H_0^1(U)}.$$

Note that we can interpret f as a distribution. It is given by a sum of an L^2 function and first derivatives of L^2 functions:

$$f \;=\; f_0 \;-\; \sum_{i=1}^n (f_i)_{x_i}.$$

However, the functions $(f_0,f_1,\dots,f_n)$ defining f are not unique. Indeed, if $h \in H^1(U)$ then $(f_0 + h_{x_1} + \cdots + h_{x_n}, f_1 + h, f_2 + h, \dots, f_n + h)$ gives the same element of $H^{-1}(U)$.

Now by the Riesz representation theorem, there exists $v \in H_0^1(U)$ such that for every $u \in H_0^1(U)$ we have

$$\langle f,u\rangle \;=\; \int_U \left(\sum_{i=1}^n v_{x_i} u_{x_i} \;+\; vu \right) dx.$$

Moreover, v is unique and

$$\|f\|_{H_0^1(U)} \;=\; \|v\|_{H_0^1(U)}.$$

From this, we see that

$$\|f\|_{H_0^1(U)} \;=\; \inf\left\{ \left(\int_U \sum_{i=0}^n g_i^2 \, dx \right)^{1/2} \;:\; \langle f,u\rangle = \int_U \left(g_0 u \;+\; \sum_{i=1}^n g_i u_{x_i} \right) dx \;\right\}.$$

Note that in the distributional sense $f = (I - \Delta)v$.

There seems to be a contradiction. On the one hand we are saying that $H^{-1}(U)$ is given by distributions which can involve derivatives of $L^2(U)$ functions, so

$$H_0^1(U) \;<\; L^2(U) \;<\; H^{-1}(U),$$

with all these inclusions strict. On the other hand we are saying that $H^{-1}(U)$ is equal to $H_0^1(U)$. What is going on? The point here is that we have two inner products in action, the inner product on $L^2(U)$ and the inner product on $H_0^1(U)$. The inclusion $L^2(U) \subset H_0^1(U)$ is given by using the $L^2(U)$ inner product, so

$$f \;\sim\; u \to \langle f, u\rangle_{L^2(U)} \;=\; \int_U fu\,dx.$$

On the other hand the identification $H^{-1}(U) \equiv H_0^1(U)$ is given by using the $H_0^1(U)$ inner product so

$$f \;\sim\; u \to \langle f, u\rangle_{H_0^1(U)} \;=\; \int_U \left(fu + \sum_{i=1}^n f_{x_i} u_{x_i} \right) dx.$$

To get from $u \in H_0^1(U)$ to the distribution it represents with the $L^2(U)$ inner product, we have

$$u \;\to\; u - \sum_i u_{x_i x_i} \;=\; (I - \Delta)u.$$

Example. $u \in L^2([0,T], H_0^1(U))$, $f \in L^2([0,T], L^2(U))$ and we want to solve $u' = f - Lu$. Notice that the right hand side is

$$f \;+\; \sum_{i,j} a^{ij}(u_{x_i})_{x_j} \;-\; \sum_i b^i u_{x_i} \;-\; cu \;\in\; L^2([0,T], H^{-1}(U)).$$

To check that the right hand side is in fact measurable from $[0,T]$ to $H^{-1}(U)$ it is useful to have the following characterization of measurable functions.

Definition. A function $u : [0,T] \to X$ is *weakly measurable* if for each $\ell \in X^*$, the function $t \to \ell(u(t))$ is Lebesgue measurable. If X is separable, then a theorem of Pettis says that weakly measurable is equivalent to strongly measurable.

Definition. Let $u \in L^1([0,T], X)$. Then $v \in L^1([0,T], X)$ is the weak derivative of u, written $u' = v$, provided

$$\int_0^T \phi'(t)u(t)\,dt \;=\; -\int_0^T \phi(t)v(t)\,dt$$

for all scalar test functions $\phi \in C_c^\infty(0,T)$.

Theorem. *If $u \in L^2([0,T], L^2(U))$ and $u' \in L^2([0,T], H^{-1}(U))$ then (after possibly being redefined on a set of measure zero)*

$$u \in C([0,T], L^2(U)).$$

Moreover, we have a bound

$$\max_{t\in[0,T]} \|u(t)\|_{L^2(U)} \;\le\; C\left(\|u\|_{L^2([0,T],H_0^1(U))} \;+\; \|u'\|_{L^2([0,T],H^{-1}(U))} \right).$$