Lecture 3: Galerkin Method.

Theorem. *If* $u \in L^2([0,T], L^2(U))$ *and* $u' \in L^2([0,T], H^{-1}(U))$ *then (after possibly being redefined on a set of measure zero)*

$$u \in C([0,T], L^2(U)).$$

The mapping

$$t \ \to \|u(t)\|_{L^2(U)}$$

is absolutely continuous, with

$$\frac{d}{dt}\|u(t)\|^2_{L^2(U)} \ = \ 2\langle u'(t), u(t)\rangle \tag{1}$$

for almost every t *with* $0 \le t \le T$*. Moreover, we have a bound*

$$\max_{t\in[0,T]} \|u(t)\|_{L^2(U)} \ \le \ C\left(\|u\|_{L^2([0,T],H^1_0(U))} \ + \ \|u'\|_{L^2([0,T],H^{-1}(U))}\right). \tag{2}$$

Proof. There are certain constructions which work for X-valued functions just the same way as for real valued functions. Indeed, first on $[-\varepsilon, T-\varepsilon]$ define $u_\varepsilon(t) = u((t+\varepsilon)/(1+2\varepsilon/T))$ so that $\tilde{u}_\varepsilon \to u$ in $L^2([0,T], H^1_0(U))$ and similarly $u'_\varepsilon \to u'$ in $L^2([0,T], H^{-1}(U))$. Then convolve with an approximate identity to get a sequence of smooth functions converging to u. We can define a bounded extension operator to a larger interval by reflecting. That is we define

$$\tilde{u}(t) \ = \ \begin{cases} u(t) & t \in [0,T] \\ u(-t) & t \in [-T,0] \\ u(2T-t) & t \in [T,2T]. \end{cases}$$

Then defining

$$v(t) \ = \ \begin{cases} u'(t) & t \in [0,T] \\ -u'(-t) & t \in [-T,0] \\ -u'(2T-t) & t \in [T,2T], \end{cases}$$

we find that $v = \tilde{u}'$.

Now for our function u, we extend u to $[-\varepsilon, T+\varepsilon]$ and then convolve with η_ε to get u_ε with $u_\varepsilon \to u$ in $L^2([0,T], H^1_0(U))$ and $u'_\varepsilon \to u'$ in $L^2([0,T], H^{-1}(U))$. Furthermore, $u_\varepsilon(t) \to u(t)$ in $H^1_0(U)$ almost everywhere and $u'_\varepsilon(t) \to u'(t)$ almost everywhere in $H^{-1}(U)$. Write

$$(u,v) = \int_U u(x)v(x)\,dx.$$

For $u_\varepsilon \in C^\infty([0,T], H_0^1(U))$ and so we have

$$\frac{d}{dt}\|u_\varepsilon(t) - u_\delta(t)\|^2_{L^2(U)} \;=\; 2\left((u'_\varepsilon(t) - u'_\delta(t))\;,\;(u_\varepsilon(t) - u_\delta(t))\right).$$

Hence

$$\begin{aligned}\|u_\varepsilon(t) - u_\delta(t)\|^2_{L^2(U)} - \|u_\varepsilon(s) - u_\delta(s)\|^2_{L^2(U)} \;&=\; 2\int_s^t \left((u'_\varepsilon(\tau) - u'_\delta(\tau))\;,\;(u_\varepsilon(\tau) - u_\delta(\tau))\right) d\tau \\ &\le\; 2\int_s^t \left(\|u_\varepsilon(\tau) - u_\delta(\tau)\|^2_{H_0^1(U)} \;+\; \|u'_\varepsilon(\tau) - u'_\delta(\tau)\|_{H^{-1}(U)}\right) d\tau \\ &\le\; \left(\|u_\varepsilon - u_\delta\|^2_{L^2([0,T],H_0^1(U))} \;+\; \|u_\varepsilon - u_\delta\|^2_{L^2([0,T],H^{-1}(U))}\right).\end{aligned}$$

Now fixing $s \in (0,T)$ with $u_\varepsilon(s) \to u(s)$ in $L^2(U)$, we find that as $\varepsilon, \delta \to 0$, we have $u_\varepsilon(t) - u_\delta(t) \to 0$ in $L^2(U)$ uniformly in t. Hence $u \in C([0,T], L^2(U))$. Since

$$\|u_\varepsilon(t)\|^2_{L^2(U)} \;=\; \|u_\varepsilon(s)\|^2_{L^2(U)} \;+\; \int_s^t (u'_\varepsilon(\tau)\;,\;u_\varepsilon(\tau))\,d\tau,$$

in the limit as $\varepsilon \to 0$, we get

$$\|u(t)\|^2_{L^2(U)} \;=\; \|u(s)\|^2_{L^2(U)} \;+\; 2\int_s^t (u'(\tau)\;,\;u(\tau))\,d\tau.$$

From this we get (1). Also, we get

$$\|u(t)\|^2_{L^2(U)} \;-\; \|u(s)\|^2_{L^2(U)} \;\le\; \left(\|u\|^2_{L^2([0,T],H_0^1(U))} \;+\; \|u'\|^2_{L^2([0,T],H^{-1}(U))}\right).$$

However, for some s we have $\|u(s)\|^2_{L^2(U)}$ is bounded by the average value

$$\frac{1}{T}\int_0^T \|u(\tau)\|^2_{L^2(U)}\,d\tau,$$

and this is bounded by $\|u\|^2_{L^2([0,T],H_0^1(U))}/T$. Hence we get (2).

Definition. Suppose

$$u \in L^2([0,T], H_0^1(U))), \qquad\qquad u' \in L^2([0,T], H^{-1}(U)).$$

We say that u is a weak solution of

$$(*) \qquad \begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t = 0\}. \end{cases}$$

if (i) for every $v \in H_0^1(U)$,

$$\langle u', v\rangle \;+\; B_t[u,v] \;=\; (f,v),$$

where B_t is the bilinear form associated to the operator L at time t, and
(ii) $u(0) = g$, which makes sense because $u \in C([0,T], L^2(U))$.