Math 231C Partial Differential Equations

Lecture 4: Galerkin Method.

Theorem. *If $f \in L^2([0,T], L^2(U))$ and $g \in L^2(U)$ then there exists a weak solution $u \in L^2([0,T], H_0^1(U))$ to*

$$(*) \qquad \begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t = 0\}. \end{cases}$$

Proof. Recall that if $u \in L^2([0,T], H_0^1(U))$ with $u' \in L^2([0,T], H^{-1}(U))$, then u is a weak solution if

$$\langle u', v\rangle \ + \ B_t[u,v] \ = \ (f,v),$$

and

$$u(0) = g,$$

which makes sense because $u \in C([0,T], L^2(U))$.

(**Galerkin Approximation**.) Choose an orthonormal base w_k for $L^2(U)$ which is orthogonal in $H_0^1(U)$. We seek a solution

$$u_m(t) \ := \ \sum_{k=1}^{m} c_m^k(t) w_k$$

to the projected equation

$$\langle u', w_k\rangle \ + \ B_t[u, w_k] \ = \ (f, w_k), \qquad 0 \le t \le T, \qquad k = 1, 2, \ldots, m,$$

with

$$(u_m(0), w_k) \ = \ (g, w_k), \qquad k = 1, 2, \ldots, m.$$

This is a standard system of ODEs. To see this, write out these equations in terms of the coefficients.

$$(c_m^k)'(t) \ + \ \sum_{\ell} c_m^\ell(t) B_t[w_\ell, w_k] \ = \ (f(t), w_k),$$

$$c_m^k(0) \ = \ (g, w_k).$$

By the standard existence theory, we get a solution $(c_m^1(t), \ldots, c_m^m(t))$. Of course we must pay attention to the coefficients. We assume $f \in L^2([0,T], L^2(U))$, so $(f, w_k) \in L^2([0,T])$. We also assume $a^{i,j}$, b^i, $c \in L^2([0,T], L^\infty(U))$, so $B_t[w_\ell, w_k] \in L^2([0,T])$.

Theorem. *(***Energy estimates***.) We have the bound*

$$\|u_m\|_{C([0,T],L^2(U))} \ + \ \|u_m\|_{L^2([0,T],H_0^1(U))} \ + \ \|u_m'\|_{L^2([0,T],H^{-1}(U))}$$
$$\le \ C(\|f\|_{L^2([0,T],L^2(U)} \ + \ \|g\|_{L^2(U)}),$$

where C is independent of m.

Proof. We follow the proof in 7.1.2.b.