**Lecture 5: Galerkin approximation continued.**

Last time: To find a weak solution to

$$(*) \qquad \begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t=0\}, \end{cases}$$

we chose

$$u_m = \sum_{k=1}^{m} c_m^k w_k$$

such that for $v \in W_m = \text{span}(w_1, \dots, w_m)$,

$$(u_m', v) + B_t[u_m, v] = (f, v),$$

and

$$(u_m(0), v) = (g, v).$$

We showed the **energy estimate**

$$\|u_m\|_{C([0,T],L^2(U))} + \|u_m\|_{L^2([0,T],H_0^1(U))} \leq C(\|f\|_{L^2([0,T],L^2(U)} + \|g\|_{L^2(U)}),$$

The final energy estimate we will need is

$$\|u_m'\|_{L^2([0,T],H^{-1}(U))} \leq C(\|f\|_{L^2([0,T],L^2(U)} + \|g\|_{L^2(U)}).$$

To see this, for $v \in H_0^1(U)$ we can write $v = v^1 + v^2$ with $v^1$ in the span of $w_1, \dots, w_m$ and $v^2$ perpendicular to this span. Then

$$(u_m', v) + B_t[u_m, v] = (f, v_1),$$

so taking $\|v\|_{H_0^1(U)} \leq 1$, we get

$$\|u_m'\|^2_{H^{-1}(U)} \leq C(\|u_m\|^2_{h_0^1(U)} + \|f\|^2_{L^2(U)}).$$

Combining this with the bounds we already have, we get the bound we want for $u_m'$.

**The weak solution of (*).** Choose a subsequence $u_{m_\ell}$ with

$$\begin{cases} u_{m_\ell} \to u & \text{weakly in } L^2([0,T], H_0^1(U)) \\ u_{m_\ell}' \to v & \text{weakly in } L^2([0,T], H^{-1}(U)). \end{cases}$$

Then the weak derivative $u'$ exists and equals $v$. Indeed, choose $w \in W_k$, and $\phi \in C_c^\infty(0,T)$. Then $\phi(t)w(x) \in L^2([0,T], H_0^1(U))$ and so

$$\left\langle \int_0^T v(t), \phi(t)\,dt\,,\, w \right\rangle \;=\; \int_0^T \langle v(t), \phi(t)w\rangle\,dt \;=\; \lim_{\ell\to\infty} \int_0^T (u'_{m_\ell}(t), \phi(t)w)\,dt$$
$$= \lim_{\ell\to\infty} \left( \int_0^T u'_{m_\ell}(t)\phi(t)\,dt\,,\, w \right) \;=\; \lim_{\ell\to\infty} \left( -\int_0^T u_{m_\ell}(t)\phi'(t)\,dt\,,\, w \right)$$
$$= \lim_{\ell\to\infty} -\int_0^T (u_{m_\ell}, \phi'(t)w)\,dt \;=\; -\int_0^T (u(t), \phi'(t)w)\,dt \;=\; -\left( \int_0^T u\phi'(t)\,dt\,,\, w \right).$$

Since this is true for all $w \in W_k$ and $\cup_k W_k$ is dense in $H_0^1(U)$, we see that

$$\int_0^T v(t)\phi(t)\,dt \;=\; -\int_0^T u(t)\phi'(t)\,dt$$

for every $\phi \in C_c^\infty(0,T)$ which says precisely that $u' = v$.

We must show that $u$ is a weak solution of (*). For $v \in W_\ell$ and $m \geq \ell$ we have

$$(u'_m, v) \;+\; B_t[u_m, v] \;=\; (f, v).$$

Multiply by $\phi(t) \in C_c^\infty(0,T)$ and integrate to get

$$\int_0^T (u'_m(t), \phi(t)v)\,dt \;+\; \int_0^T B_t[u_m, \phi(t)v]\,dt \;=\; \int_0^T (f(t), \phi(t)v)\,dt.$$

Taking the limit as $m = m_\ell \to \infty$ we get

$$\int_0^T \langle u'(t), \phi(t)v\rangle\,dt \;+\; \int_0^T B_t[u, \phi(t)v]\,dt \;=\; \int_0^T (f(t), \phi(t)v)\,dt.$$

Now since $\cup_k W_k$ is dense in $H_0^1(U)$, we find that this holds for every $v \in H_0^1(U)$. We can write this as

$$\int_0^T \left(\langle u'(t), v\rangle + B_t[u,v] \;-\; (f(t), v)\right)\phi(t)\,dt \;=\; 0.$$

But since $C_0^\infty(0,T)$ is dense in $L^1(0,T)$ we get

$$\langle u'(t), v\rangle + B_t[u,v] \;-\; (f(t), v) \;=\; 0$$

almost everywhere. This says that $u$ is a weak solution of (*).

It remains to show $u(0) = g$. Again, take $v \in W_k$ and consider $\phi(t)v$ with $\phi \in C_c^\infty[0,T)$ and $\phi(0) = 1$. Then for $m \geq k$,

$$\int_0^T (u_m'(t), \phi(t)v) \;+\; (u_m(t), \phi'(t)v)\, dT \;=\; -(u_m(0), v(0)) \;=\; -(g, v).$$

However, taking the limit as $m = m_\ell \to \infty$ this gives

$$(***) \qquad \int_0^T \langle u'(t), \phi(t)v\rangle \;+\; (u, \phi'(t)v)\, dt \;=\; (g, v).$$

But the left hand side is $(u(0), v)$. To see this, just note that choose $\psi_\varepsilon \in C_c^\infty[0,T)$ supported in $[0,\varepsilon)$ with $\psi_\varepsilon(t) = \phi(t)$ for $t \in [0, \varepsilon/2)$ and $\psi_\varepsilon' \leq 0$. Then $0 \leq \psi_\varepsilon \leq 1$ and by the definition of weak derivatives,

$$\int_0^T u'(t)(\phi(t) - \psi_\varepsilon(t))\, dt \;+\; \int_0^T u(t)(\phi'(t) - \psi_\varepsilon'(t))\, dt \;=\; 0.$$

Hence

$$\int_0^T (u'(t)\phi(t) \;+\; u(t)\phi'(t))\; dt \;=\; \int_0^T (u'(t)\psi_\varepsilon(t) \;+\; u(t)\psi_\varepsilon'(t))\; dt.$$

But

$$\left\| \int_0^T u'(t)\psi_\varepsilon((t)\, dt \right\|_{H^{-1}(U)} \;\leq\; \int_0^\varepsilon \|u'(t)\|_{H^{-1}(U)}\, dt \;\to\; 0$$

as $\varepsilon \to 0$, while since $u \in C([0,T], L^2(U))$ we have

$$\int_0^T u(t)\psi_\varepsilon'(t)\, dt \;\to\; -u(0)$$

as $\varepsilon \to 0$.

Hence from (***) we get

$$(u(0), v) \;=\; (g, v).$$

Since $\cup_\ell W_k$ is dense in $L^2(U)$ we get $u(0) = g$.