MATH 231C PARTIAL DIFFERENTIAL EQUATIONS

Lecture 6: Problem session.

1. Suppose $u \in L^2([0,T], H_0^1(U))$. Exactly what does it mean that the weak derivative u' exists in $L^2([0,T], H^{-1}(U))$?

2. Show that if $u \in C([0,T], L^2(U))$ and $u' \in L^2([0,T], H^{-1}(U))$ and $\phi \in C^\infty([0,T])$ has $\phi(T) = 0$, then

$$\int_0^T (\phi u' + \phi' u)\, dt \;+\; \phi(0)u(0) \;=\; 0.$$

3. For a smooth function $g : U \to \mathbb{R}$, write down the norms of g in the spaces $L^2(U)$, $H_0^1(U)$ and $H^{-1}(U)$. (Here, g defines the element of $H^{-1}(U)$ by using the $L^2(U)$ inner product.)

4. Suppose that $w_1, w_2, \dots$ is an orthonormal basis for $L^2(U)$ consisting of eigenfunctions for the Laplacian with Dirichlet boundary conditions, that is

$$\begin{cases} -\Delta w_j \;=\; \lambda_j w_j, & \text{weakly in } U \\ w_j = 0 & \text{on } \partial U. \end{cases}$$

Show that

$$\|w_j\|^2_{H_0^1(U)} \;=\; \lambda_j \|w_j\|^2_{L^2(U)}.$$

For a function $f \in H_0^1(U)$ set

$$\hat{f}_j \;=\; \int_U w_j f\, dx.$$

Write the norms $\|f\|^2_{L^2(U)}$ and $\|f\|^2_{H_0^1(U)}$ in terms of the coefficients $\hat{f}_j$. What is the orthogonal projection in $L^2(U)$ of f onto the span of $w_1, \dots, w_m$? How about if you take the projection in the space $H_0^1(U)$?