

**Lecture 7: Regularity.**

Last week we found a weak solution to the equation

$$(*) \quad \begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0, T] \\ u = g & \text{on } U \times \{t = 0\}, \end{cases}$$

where

$$Lu = - \sum_{i,j=1}^n (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^n b^i u_{x_i} + cu.$$

The precise conditions were as follows:

$$\begin{aligned} a^{ij}, b^i, c &\in L^\infty(U_T). \\ f &\in L^2([0, T], L^2(U)), \quad g \in L^2(U). \\ u &\in L^2([0, T], H_0^1(U)) \cap C([0, T], L^2(U)), \\ u_t &\in L^2([0, T], H^{-1}(U)). \end{aligned}$$

To see that the solution is unique, if we had two solutions then the difference would satisfy

$$\begin{cases} u_t + Lu = 0 & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0, T] \\ u = 0 & \text{on } U \times \{t = 0\}, \end{cases}$$

Then

$$\frac{d}{dt} \|u\|_{L^2(U)}^2 = 2(u', u) = -2B[u, u] \leq \gamma \|u\|_{L^2(U)}^2.$$

(we showed that the first equality holds by approximating  $u$  by  $\eta_\varepsilon * u$  where  $\eta_\varepsilon$  is an approximate identity) so

$$\frac{d}{dt} \left( e^{-\gamma t} \|u\|_{L^2(U)}^2 \right) \leq 0.$$

But then since  $e^{-\gamma t} \|u\|_{L^2(U)}^2$  is non-negative and equals zero at  $t = 0$ , we see that it is identically zero so  $u \equiv 0$ .

This week we will prove regularity for the solution  $u$ .

**Theorem.** *With the following assumptions on  $L, f, g$  we conclude the given regularity of  $u$ . Moreover the given norms on  $u$  are bounded by a multiple of the sum of the given norms on  $f$  and  $g$ .*

$$\begin{aligned}
& a^{ij}, b^i, c \in C^\infty(\bar{U}) \text{ are independent of } t. \\
& f \in L^2([0, T], L^2(U)), \quad g \in H_0^1(U). \\
& u \in L^2([0, T], H^2(U)) \cap L^\infty([0, T], H_0^1(U)), \\
& u' \in L^2([0, T], L^2(U)).
\end{aligned}$$

**Example.** If  $u$  is a smooth solution of

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0, T] \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

and if  $u$  decays sufficiently rapidly as  $|x| \rightarrow \infty$ , then

$$\begin{aligned}
\int_{\mathbb{R}^n} f^2 dx &= \int_{\mathbb{R}^n} (u_t - \Delta u)^2 dx \\
&= \int_{\mathbb{R}^n} u_t^2 + (\Delta u)^2 - 2u_t \Delta u dx \\
&= \int_{\mathbb{R}^n} u_t^2 dx + \int_{\mathbb{R}^n} |D^2 u|^2 dx + \frac{d}{dt} \int_{\mathbb{R}^n} |Du|^2 dx.
\end{aligned}$$

Hence

$$\int_{\mathbb{R}^n} |Du(x, t)|^2 dx + \int_0^t \int_{\mathbb{R}^n} u_t^2 dx dt + \int_0^t \int_{\mathbb{R}^n} |D^2 u|^2 dx dt = \int_0^t \int_{\mathbb{R}^n} f^2 dx dt + \int_{\mathbb{R}^n} |Dg|^2 dx.$$

We get

$$\|Du\|_{L^\infty([0, T], L^2(U))}^2 + \|u_t\|_{L^2(U)}^2 + \|D^2 u\|_{L^2([0, T], L^2(U))}^2 \leq 2(\|f\|_{L^2([0, T], L^2(U))}^2 + \|g\|_{H_0^1(U)}^2).$$

**Proof of the Theorem.** We use our approximate solutions

$$u_m(x, t) = \sum_{k=1}^m c_m^k(t) w_k(x)$$

which satisfy the projected weak equation

$$\begin{cases} (u'_m, v) + B[u_m, v] = (f, v) & v \in W_m \\ (u_m(0), v) = (g, v) & v \in W_m. \end{cases}$$

Then

$$(u'_m, u'_m) + B[u_m, u'_m] = (f, u'_m).$$

Set

$$A[u, v] = \sum_{i,j} a^{ij} u_{x_i} v_{x_j}.$$

Then

$$B[u_m, u'_m] = \frac{1}{2} \frac{d}{dt} A[u_m, u_m] + \sum_{i=1}^n b^i(u_m)_{x_i} u'_m + cu_m u'_m.$$

Hence

$$\begin{aligned} (u'_m, u'_m) + \frac{1}{2} \frac{d}{dt} A[u_m, u_m] &= (f, u'_m) - \sum_{i=1}^n (b^i(u_m)_{x_i}, u'_m) - (cu_m, u'_m) \\ &\leq \frac{1}{\varepsilon} \|f\|_{L^2(U)}^2 + \varepsilon \|u'_m\|_{L^2(U)}^2 + \frac{C}{\varepsilon} \|u_m\|_{H^1(U)}^2 + \varepsilon \|u'_m\|_{L^2(U)}^2. \end{aligned}$$

Choosing  $\varepsilon = 1/4$  we get

$$(u'_m, u'_m) + \frac{1}{2} \frac{d}{dt} A[u_m, u_m] \leq C \left( \|f\|_{L^2(U)}^2 + \|u_m\|_{H^1_0(U)}^2 \right).$$

Integrating from 0 to  $t$  gives

$$\begin{aligned} \|u'_m\|_{L^2([0,T],L^2(U))}^2 + \sup_{t \in [0,T]} A[u_m, u_m] \\ \leq C' \left( \|f\|_{L^2([0,T],L^2(U))}^2 + \|u_m\|_{L^2([0,T],H^1_0(U))}^2 \right) + A[u_m(\cdot, 0), u_m(\cdot, 0)]. \end{aligned}$$

Now

$$\sup_{t \in [0,T]} A[u_m, u_m] \sim \|u_m\|_{L^\infty([0,T],H^1_0(U))}^2,$$

meaning that each of these two quantities is bounded by a multiple of the other.

Hence we get

$$\begin{aligned} \|u'_m\|_{L^2([0,T],L^2(U))}^2 + \|u_m\|_{L^\infty([0,T],H^1_0(U))}^2 \\ \leq C \left( \|f\|_{L^2([0,T],L^2(U))}^2 + \|u_m\|_{L^2([0,T],H^1_0(U))}^2 + \|u_m(0)\|_{H^1_0(U)}^2 \right). \end{aligned}$$