Lecture 8: Regularity.

$$(*)\qquad \begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t=0\}, \end{cases}$$

where

$$Lu \;=\; -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} \;+\; \sum_{i=1}^{n} b^i u_{x_i} \;+\; cu.$$

We showed existence of weak solutions. We are in the middle of showing that if

$$\begin{aligned} &a^{ij},\ b^i,\ c\ \in C^\infty(\bar U) \text{ are independent of } t. \\ &f \in L^2([0,T], L^2(U)), \qquad g \in H_0^1(U), \end{aligned}$$

then we have

$$\begin{aligned} &u \in L^2([0,T], H^2(U)) \ \cap\ L^\infty([0,T], H_0^1(U)), \\ &u' \in L^2([0,T], L^2(U)), \end{aligned}$$

with bounds. (Later we will replace the L^∞ by C.) So far we took

$$u_m \;=\; \sum_{k=1}^{m} c_m^k(t) w_k$$

solving the projected weak equation. The projected weak equation with test function u_m' is

$$(u_m', u_m') \;+\; B_t[u_m, u_m'] \;=\; (f, u_m').$$

From this we deduce

$$\begin{aligned} &\|u_m'\|^2_{L^2([0,T],L^2(U))} \;+\; \sup_{t\in[0,T]} \|u_m(t)\|^2_{H_0^1(U))} \\ &\qquad \le\; C\left(\|f\|^2_{L^2([0,T],L^2(U))} \;+\; \|u_m\|^2_{L^2([0,T],H_0^1(U))} \;+\; \|u_m(0)\|^2_{H_0^1(U)}\right). \end{aligned}$$

Also, by last week's energy inequalities we have

$$\|u_m\|_{L^2([0,T],H_0^1(U)} \;\le\; C\left(\|f\|_{L^2([0,T],L^2(U))} \;+\; \|g\|_{L^2(U)}\right).$$

Moreover, since $u_m(0)$ is equal to the projection of g onto W_m, we see that

$$\|Du_m(0)\|_{L^2(U)} \;\le\; \|u_m(0)\|_{H_0^1(U)} \;\le\; \|g\|_{H_0^1(U)}.$$

Putting this together we have
(**)

$$\|u'_m\|^2_{L^2([0,T],L^2(U))} \ + \ \sup_{t\in[0,T]} \|u_m\|^2_{H^1_0(U))} \ \leq \ C\left(\|f\|^2_{L^2([0,T],L^2(U))} \ + \ \|g\|^2_{L^2(U)}\right).$$

Now we take a subsequence

$$\begin{cases} u_{m_\ell} \to u & \text{weakly in } L^2([0,T],H^1_0(U)) \\ u'_{m_\ell} \to u' & \text{weakly in } L^2([0,T],L^2(U)). \end{cases}$$

Then u is the weak solution of (*), and we get immediately the stated bound on u'. For the L^∞ bound on u, we have more generally that if H is a Hilbert space and $u_\ell \to u$ weakly in $L^2([0,T],H)$, and we have a uniform bound

$$\|u_\ell(t)\| \ \leq \ C,$$

then

$$\|u(t)\| \ \leq \ C.$$

To see this, we will use the fact that the dual space of $L^1([0,T])$ is $L^\infty([0,T])$. Write (,) for the inner product on H, and take $v \in H$ and $\phi \in L^2[0,T]$. Then

$$\int_0^T (u(t),v)\phi(t)\,dt \ = \ \lim_{\ell\to\infty}\int_0^T\int_U (u_\ell(t),v)\phi(t)\,dt \ \leq \ C\|v\|\|\phi\|_{L^1([0,T])}.$$

Since $L^2([0,T])$ is dense in $L^1([0,T])$ we see that

$$(u(t),v) \ \leq \ C\|v\|,$$

so $\|u(t)\| \leq \ C$. Hence we have shown that we can pass to the limit in (**) to get the stated bound for $\sup_{t\in[0,T]}\|u\|_{H^1_0(U))}$.

It remains to bound $\|u\|_{L^2([0,T],H^2(U))}$, but for almost every t and every $v \in H^1_0(U)$, we have

$$(u'(t),v) \ + \ B[u(t),v] \ = \ (f(t),v),$$

so

$$B[u(t),v] \ = \ (f(t) - u'(t),v).$$

But $f(t) - u'(t) \in L^2(U)$ for almost every t, and for such t we have that $u(t)$ is a weak solution to the equation $Lu(t) = (f(t) - u'(t)) \in L^2(U)$ and so by the regularity from the previous section (using difference quotients) we have $u(t) \in H^2(U)$ and

$$\|u(t)\|^2_{H^2(U)} \ \leq \ C(\|f(t)\|^2_{L^2(U)} \ + \ \|u'(t)\|^2_{L^2(U)} \ + \ \|u\|^2_{L^2(U)}).$$

Integrating this from $t = 0$ to T and using the bound already established for u' gives the stated inequality.

Sobolev Norms of Projections onto W_k. Recall that $w_1, w_2, \dots$ are orthonormal in $L^2(U)$ and orthogonal in $H_0^1(U)$. If $u \in H_0^1(U)$, then

$$u = \sum_{k=1}^{\infty} (w_k, u) w_k,$$

with convergence both in $L^2(U)$ and $H_0^1(U)$. To see that these are the correct coefficients in $H_0^1(U)$, note that for $u \in H_0^1(U)$,

$$(w_k, u)_{H_0^1(U)} = (w_k, u) + \sum_{i=1}^{n} ((w_k)_{x_i}, u_{x_i}) = ((1 - \Delta) w_k, u) = (1 + \lambda_k)(w_k, u),$$

where λ_k are the eigenvalues, $-\Delta w_k = \lambda_k w_k$. Hence applying this to $u = w_k$ we see that $w_k/(1+\lambda_k)^{1/2}$ are orthonormal in $H_0^1(U)$. But then in $H_0^1(U)$ we have

$$u = \sum_k \left(\frac{w_k}{(1+\lambda_k)^{1/2}}, u \right) \frac{w_k}{(1+\lambda_k)^{1/2}} = \sum_k (w_k, u) w_k.$$

We conclude that the orthogonal projections of u onto W_m in the spaces $L^2(U)$ and $H_0^1(U)$ are the same. Defining

$$u_m = \sum_{k=1}^{m} (w_k, u) w_k,$$

we see that

$$\|u_m\|_{L^2(U)} \leq \|u\|_{L^2(U)}, \qquad \|u_m\|_{H_0^1(U)} \leq \|u\|_{H_0^1(U)}.$$

For H^2 bounds however, the situation is more subtle.

Lemma. *For $u \in H_0^1(U)$ here exists $C_k = C_k(U)$ such that*

$$\|u_m\|_{H^k(U)} \leq C_k \|u\|_{H^k(U)}.$$

Proof. For $k = 2$ we have that by the elliptic theorem (proved with difference quotients),

$$\begin{aligned} \|u_m\|_{H^2(U)}^2 &\leq C \|\Delta u_m\|_{L^2(U)}^2 = C(\Delta^2 u_m, u_m) \\ &\leq C(\Delta^2 u_m, u) = C(\Delta u_m, \Delta u) \leq C \|u_m\|_{H^2(U)} \|u\|_{H^2(U)}, \end{aligned}$$

Dividing by $\|u_m\|_{H^2(U)}$ gives the result. For higher k, one uses the higher elliptic regularity theorem.