MATH 231C PARTIAL DIFFERENTIAL EQUATIONS

Lecture 9: Slightly Higher Regularity. .

Recall u is the weak solution to

$$(*)\qquad \begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t=0\}, \end{cases}$$

where

$$Lu \;=\; -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} \;+\; \sum_{i=1}^{n} b^i u_{x_i} \;+\; cu.$$

So for almost every t,

$$\begin{aligned} &(u_t, v) + B[u,v] = (f,v), &&\text{for all } v \in H_0^1(U), \\ &u(0) = g. \end{aligned}$$

Theorem.

$a^{ij},\ b^i,\ c\ \in C^\infty(\bar{U})$ are independent of t.
$f \in L^2([0,T], L^2(U)),\qquad f' \in L^2([0,T], L^2(U)),\qquad g \in H^2(U) \cap H_0^1(U),$

then we have

$$\begin{aligned} &u \in L^\infty([0,T], H^2(U)) \;\cap\; L^\infty([0,T], H_0^1(U)), \\ &u' \in L^2([0,T], H_0^1(U)) \cap C([0,T], L^2(U)), \\ &u'' \in L^2([0,T], H^{-1}(U)). \end{aligned}$$

with bounds.

Idea of the Proof. What equation should u' satisfy? Differentiating (*) and setting $\tilde{u} = u'$ we get

$$(**)\qquad \begin{cases} \tilde{u}_t + L\tilde{u} = f_t & \text{in } U \times [0,T), \\ \tilde{u} = 0 & \text{on } \partial U \times [0,T], \\ \tilde{u} = f - Lg & \text{on } U \times \{t=0\}. \end{cases}$$

By the existence theorem we get a unique solution $\tilde{u}$ to this problem, but is this solution really equal to u'?

Proof of the Theorem. Set $\tilde{u} = u'$. Then recalling that

$$u_m \;=\; \sum_{k=1}^{m} c_m^k(t) w_k$$

is constructed to solve

$$(1) \qquad (u_m', v) + B[u_m, v] = (f, v), \qquad v \in W_m,$$

$$(2) \qquad (u_m(0), v) = (g, v), \qquad v \in W_m.$$

Let g_m be the orthogonal projection of g onto W_m, that is

$$g_m = \sum_{k=1}^{m} (g, w_k) w_k.$$

Then

$$(3) \qquad (\tilde{u}_m', v) + B[\tilde{u}_m, v] = (f', v), \qquad \text{for almost every } t,$$

$$(4) \qquad (\tilde{u}_m(0), v) = (f(0) - Lg_m, v), \qquad v \in W_m.$$

We get (3) by taking the weak derivative of (1). To get (2), evaluate (1) at $t = 0$ to get

$$(u_m'(0), v) + B[g_m, v] = (f(0), v).$$

But

$$\|g_m\|_{H^2(U)} \le C\|g\|_{H^2(U)},$$

so $B[g_m, v] = (Lg_m, v)$, and we get (4). But $f(0) - Lg_m \in L^2$ with

$$\|f(0) - Lg_m\|_{L^2(U)} \le C(\|f\|_{H^1([0,T],L^2(U)} + \|g\|_{H^2(U)}).$$

Applying the energy inequalities in the existence proof to the functions $\tilde{u}_m = u_m'$, we get

$$\max_{0 \le t \le T} \|u_m'(t)\|_{L^2(U)} + \|u_m'\|_{L^2([0,T],H_0^1(U))} + \|u_m''\|_{L^2([0,T],H^{-1}(U))} \le C(\|f\|_{H^1([0,T],L^2(U))} + \|g\|_{H^2(U)}).$$

Now take

$$\begin{cases} u_m \to m & \text{weakly in } L^2([0,T], H^2(U)), \\ u_m' \to u' & \text{weakly in } L^2([0,T], H_0^1(U)), \\ u_m'' \to u'' & \text{weakly in } L^2([0,T], H^{-1}(U)). \end{cases}$$

The bounds persist when we take limits and we get all the stated bounds except $u \in L^2([0,T], H^2(U))$. For this we know that for almost every t,

$$(u', v) + B[u, v] = (f, v),$$

but since $u(t) \in H^2(U)$ almost everywhere we in fact have

$$u' + Lu = f \qquad \text{for almost every } t.$$

But then for almost every t,

$$\|Lu(t)\|_{L^2(U)} \;\leq\; \|f(t)\|_{L^2(U)} \;+\; \|u'(t)\|_{L^2(U)},$$

and so by elliptic regularity,

$$\|u(t)\|_{H^2(U)} \;\leq\; C(\|f(t)\|_{L^2(U)} \;+\; \|u'(t)\|_{L^2(U)}).$$

But the right hand side is uniformly bounded in t, which completes the proof.

Higher Regularity. Consider the equation

$$(*) \qquad \begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t = 0\}. \end{cases}$$

We notice that there is an intersection between the two regions where we are prescribing boundary conditions, namely this intersection is $\partial U \times \{t = 0\}$. For the boundary conditions to be compatible we therefore need g to vanish on ∂U, as indeed we assume with the condition $g \in H_0^1(U)$ which we needed to get the improved regularity. To understand higher regularity, we first notice that this is not the only compatibility condition for the data. Indeed, since we require $u = 0$ on $\partial U \times [0, T)$, if $u \in C^{2k,k}[U \times [0, T))$ then this places the condition

$$g_k(x) \;:=\; \partial_t^k u(x,0) \;=\; 0, \qquad \text{when } x \in \partial U.$$

However, $g_0 = g$ and we can express g_k in terms of g_{k-1}. Indeed,

$$(**) \qquad u_t \;=\; f - Lu,$$

so

$$g_1 \;=\; f - Lg_0.$$

and differentiating (**)

$$\partial_t^k u \;=\; \partial_t^{k-1} f^{k-1} \;-\; L\left(\partial_t^k u\right).$$

so

$$(***) \qquad g_k(x) \;=\; \partial_t^{k-1} f(x,0) \;-\; Lg_{k-1}(x).$$

The compatibility condition between the boundary conditions is then

$$g_0, g_1, \ldots, g_k = 0 \qquad \text{on } \partial U.$$

Theorem. *(Higher Regularity). Define $g_0 = g$ and define g_k inductively by (***). Assume*

$$a^{ij},\; b^i,\; c \;\in C^\infty(\bar{U}) \text{ are independent of } t.$$

$$\frac{d^k f}{dt^k} \in L^2([0,T], H^{2m-2k}(U)),\; (k = 0, \ldots, m), \qquad g \in H^{2m+1}(U),$$

$$g_0, \ldots, g_m \in H_0^1(U).$$

then the solution of (*) satisfies

$$\frac{d^k u}{dt^k} \in L^2([0,T], H^{2m+2-2k}(U)), \qquad (k = 0, \ldots, m+1)$$

with bounds.