

Title: Conformally flat submanifolds in spheres and soliton equations

Abstract

A hypersurface in S^{n+1} is conformally flat if there exists a flat metric in the conformal class of the induced metric. When $n > 3$, this condition is equivalent to the Weyl tensor being zero and when $n = 3$ it is equivalent to the Schouten tensor S being a Codazzi tensor, i.e., ∇S is a symmetric 3-tensor. É. Cartan proved that if $n > 3$ a conformally flat hypersurface in S^{n+1} has at most two distinct principal curvatures, and hence is given by a one-parameter family of $(n - 1)$ -spheres. However, the $n = 3$ case is more complicated, and in fact it is governed by a soliton equation.

Soliton equations are non-linear hyperbolic equations, and are completely integrable Hamiltonian PDEs with respect to two symplectic structures. Characteristic properties of soliton equations include the existence of an infinite family of explicit soliton solutions, a loop group action on the space of solutions, and a scattering and inverse scattering theory to solve the Cauchy problem with rapidly decaying initial data. Examples are, KdV, Non-linear Schrödinger, and the Sine-Gordon equation.

In a joint work with Neil Donaldson, we show that the Gauss-Codazzi equations for conformally flat hypersurfaces in S^4 is a soliton equation so that techniques from soliton theory can be used to study them. In particular, we construct (i) infinitely many families of explicit conformally flat hypersurfaces in S^4 , (ii) infinitely many new conformally flat hypersurfaces in S^4 from a given one, and (iii) a loop group action of the moduli space. We also show that a conformally flat hypersurface in S^4 is determined by four functions of one variable and a null vector in $\mathbb{R}^{2,1}$. These results are generalized to conformally flat n -submanifolds in S^{2n-2} with flat and non-degenerate normal bundle.

Reference: arXiv:0803.2754, Conformally flat submanifolds in spheres and integrable systems, by N. Donaldson and C-L Terng