11. Show that if \( u(x,t) = (f \ast H_t)(x) \) where \( H_t \) is the heat kernel, and \( f \) is Riemann integrable, then \( \int_0^1 |u(x,t) - f(x)|^2 dx \to 0 \) as \( t \to 0 \).

By (8) On P.119 of the textbook, we know that \( u(x,t) = \sum_{n=-\infty}^{\infty} a_n e^{-4\pi^2 n^2 t} e^{2\pi i n x} \) where \( a_n = \hat{f}(n) \)

By Mean-Square Convergence of the Fourier Series, we know that for any \( \epsilon > 0 \), there exists \( N_1 > 0 \) s.t. for \( k \geq N_1 \), we have
\[
\int_0^1 |f(x) - \sum_{n=-\infty}^{k} a_n e^{2\pi i n x}|^2 dx < \epsilon \tag{1}
\]
Indeed, by Parseval identity, we know that \( \sum_{n=-\infty}^{\infty} |a_n|^2 \) converges. Hence for the same \( \epsilon > 0 \), there exists \( N_2 > 0 \) s.t. \( \sum_{|n| \geq k} |a_n|^2 dx < \epsilon \) for \( k \geq N_2 \tag{2} \)

Also, we know that there exists \( A > 0 \), \( a_n < A \) for all \( n \in \mathbb{R} \) \tag{3}

for any \( \epsilon > 0 \) take \( T \) with \( |(e^{-4\pi^2 n^2 T} - 1) | < \frac{\epsilon}{N_A} \) where \( N = \max \{ N_1, N_2 \} \), we have for all \( 0 < t < T \),
\[
\left( \int_0^1 |u(x,t) - f(x)|^2 dx \right)^{\frac{1}{2}} = \left( \int_0^1 \left| \sum_{n=-\infty}^{\infty} a_n e^{-4\pi^2 n^2 t} e^{2\pi i n x} - f \right|^2 dx \right)^{\frac{1}{2}}
\leq \left( \int_0^1 \left| \sum_{n=-\infty}^{N} a_n e^{-4\pi^2 n^2 t} e^{2\pi i n x} - \sum_{n=-N}^{N} a_n e^{2\pi i n x} \right|^2 dx \right)^{\frac{1}{2}}
\leq \left( \int_0^1 \left| \sum_{n=-N}^{N} a_n e^{-4\pi^2 n^2 t} e^{2\pi i n x} \right|^2 \right)^{\frac{1}{2}} + \int_0^1 \left| \sum_{n=-N}^{N} a_n e^{2\pi i n x} - f \right|^2 dx \frac{1}{2}
\leq \left( \int_0^1 \left| \sum_{|n| \geq N} a_n e^{-4\pi^2 n^2 t} e^{2\pi i n x} \right|^2 \right)^{\frac{1}{2}} + \left( \int_0^1 \left| \sum_{n=-N}^{N} a_n e^{2\pi i n x} - f \right|^2 dx \right)^{\frac{1}{2}}
\leq \left( \int_0^1 \sum_{|n| \geq k} |a_n|^2 dx \right)^{\frac{1}{2}} + \epsilon \quad \text{by (1)}
\leq \sum_{n=-N}^{N} \left( \int_0^1 \left| a_n e^{-4\pi^2 n^2 t} e^{2\pi i n x} - a_n e^{2\pi i n x} \right|^2 dx \right)^{\frac{1}{2}} + 2\epsilon \quad \text{by (2)}
= \sum_{n=-N}^{N} \left( \int_0^1 \left| a_n e^{-4\pi^2 n^2 t} - 1 \right|^2 |a_n|^2 dx \right)^{\frac{1}{2}} + 2\epsilon
\leq A \sum_{n=-N}^{N} \left( \int_0^1 \left| e^{-4\pi^2 n^2 t} - 1 \right|^2 dx \right)^{\frac{1}{2}} + 2\epsilon \quad \text{by (3)}
\leq 3\epsilon
2. (a) Show that for $H \leq N$, $|S_N|^2 \leq c \frac{N}{H} \sum_{h=0}^{H} |\sum_{n=1}^{N-h} e^{2\pi i(f(n+h)-f(n))}|$

Let

$$a_n = \begin{cases} e^{2\pi i f(n)} & \text{if } 1 \leq n \leq N \\ 0 & \text{if } n \leq 0 \end{cases}$$

where $n \in \mathbb{Z}$.

Since $|a_n| = 1$, $\overline{a_n} = \frac{1}{a_n}$.

It suffices to prove $|\sum_{n \in \mathbb{Z}} a_n|^2 \leq c \frac{N}{H} \sum_{h=0}^{H} |\sum_{n=1}^{N-h} a_{n+h}\overline{a_n}|$.

First, we have $H \sum_{n \in \mathbb{Z}} a_n = \sum_{n \in \mathbb{Z}} \sum_{k=1}^{H} a_{n+k}$ as there are only finite non-zero terms. Hence,

$$H^2 |\sum_{n \in \mathbb{Z}} a_n|^2 = |\sum_{n \in \mathbb{Z}} \sum_{k=1}^{H} a_{n+k}|$$

$$= |\sum_{n=-H+1}^{N-1} \sum_{k=1}^{H} a_{n+k}|^2$$

since, for $1 \leq k \leq H$, $a_{n+k} = 0$ if $n \notin [-H+1, N-1]$

$$\leq [N - 1 - (-H + 1)] \sum_{n=-H+1}^{N-1} \sum_{k=1}^{H} |a_{n+k}|^2 \quad \text{By Cauchy-Schwarz}$$

$$\leq 2N \sum_{n=-H+1}^{N-1} |\sum_{k=1}^{H} a_{n+k}|^2$$

$$= 2N \sum_{n \in \mathbb{Z}} |\sum_{k=1}^{H} a_{n+k}|^2$$

for $|\sum_{k=1}^{H} a_{n+k}|^2$, we have

$$|\sum_{k=1}^{H} a_{n+k}|^2 = \left(\sum_{k=1}^{H} a_{n+k}\right)\left(\sum_{m=1}^{H} a_{n+m}\right)$$

$$= \left(\sum_{k=1}^{H} \overline{a_{n+k}}\right)\left(\sum_{m=1}^{H} a_{n+m}\right)$$

$$= \sum_{1 \leq k,m \leq H} a_{n+k}a_{n+m}$$

$$= \sum_{k=1}^{H} |a_n + k|^2 + \sum_{1 \leq k,m \leq H, k \neq m} \overline{a_{n+k}}a_{n+m}$$

the sum is split into two parts $k = m$ and $k \neq m$

$$= \sum_{k=1}^{H} |a_n + k|^2 + 2 \sum_{1 \leq k<m \leq H, k \neq m} Re(\overline{a_{n+k}}a_{n+m})$$

Hence, $\sum_{n \in \mathbb{Z}} |\sum_{k=1}^{H} a_{n+k}|^2 = \sum_{n \in \mathbb{Z}} \sum_{k=1}^{H} |a_{n+k}|^2 + 2 \sum_{1 \leq k<m \leq H, k \neq m} Re(\overline{a_{n+k}}a_{n+m})$
Concerning the first part $\sum_{n \in \mathbb{Z}} \sum_{k=1}^{H} |a_n + k|^2$, we have

$$\sum_{n \in \mathbb{Z}} \sum_{k=1}^{H} |a_n + k|^2 = \sum_{k=1}^{H} \sum_{n \in \mathbb{Z}} |a_n + k|^2$$

since there are only finite non-zero terms

$$= H \sum_{n=1}^{N} |a_n|^2$$

Concerning the second part, $\sum_{n \in \mathbb{Z}} \sum_{1 \leq k < m \leq H, k \neq m} \overline{a_{n+k}} a_{n+m}$,

$$\sum_{n \in \mathbb{Z}} \sum_{1 \leq k < m \leq H} \overline{a_{n+k}} a_{n+m} = \sum_{n \in \mathbb{Z}} \sum_{1 \leq k < m \leq H} \overline{a(n+k)} a(n+k)+(m-k)$$

since there are only finite non-zero terms

$$= H \sum_{w=1}^{H-1} \sum_{k=1}^{H-1} \sum_{n \in \mathbb{Z}} \overline{a_{n+k}} a_{n+k+w}$$

where $(m, k)$ entry can be viewed as upper triangular block of $H \times H$ matrix where $m - k = w$ is each diagonal of the block which has $H - w$ elements.

$$= H \sum_{w=1}^{H-1} \sum_{m=1}^{H-1} \sum_{n \in \mathbb{Z}} \overline{a_{n+k}} a_{n+k+w} N_w$$

where $N_w$ is the combination of $n + k = m$ for fixed $w$, the combination depends on $k$ which is $H - w$

$$\leq H \sum_{w=1}^{H-1} \sum_{m=1}^{H-1} \overline{a_{m}} a_{m+w} N_w$$

Hence,

$$H^2 \sum_{n \in \mathbb{Z}} |a_n|^2 \leq 2N \sum_{n \in \mathbb{Z}} \sum_{k=1}^{H} |a_n + k|^2$$

$$\leq 2N \sum_{n \in \mathbb{Z}} \sum_{k=1}^{H} |a_n + k|^2 + 2 \text{Re} \left( \sum_{n \in \mathbb{Z}} \sum_{1 \leq k < m \leq H, k \neq m} \overline{a_{n+k}} a_{n+m} \right)$$

$$\leq 2N \left[ H \sum_{n=1}^{N} |a_n|^2 + 2 \text{Re} \left( H \sum_{w=1}^{H-1} \sum_{m=1}^{H-1} \overline{a_{m}} a_{m+w} \right) \right]$$

$$\leq 2N \left[ H \sum_{m=1}^{N} \overline{a_{m}} a_{m} + 2H \sum_{w=1}^{H-1} \sum_{m=1}^{N-w} \overline{a_{m}} a_{m+w} \right]$$

$$\leq 2NH \sum_{w=0}^{H-1} \sum_{m=1}^{N-w} \overline{a_{m}} a_{m+w}$$
Hence, \( |\sum_{n \in \mathbb{Z}} a_n|^2 \leq 2N^2 \sum_{w=0}^{H} |\sum_{m=1}^{N-w} a_m a_{m+w}| \)

(b) Show \( < n^2 \gamma > \) is equidistributed in \([0, 1)\) whenever \( \gamma \) is irrational.

By Weyl’s criteria, it suffices to show that for any \( k \neq 0 \), \( \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i < n^2 \gamma > k} \to 0 \) as \( N \to \infty \)

First, for any \( \epsilon > 0 \), there exists \( H > 0 \) s.t. \( \frac{1}{H} \leq \epsilon \)

Then, We use the fact that \( < 2nh \gamma > \) is equidistributed for any integer \( h \), for any \( k > 0 \) there exists \( M(k,h) > 0 \) s.t.

Thus, for any \( \epsilon > 0 \), for \( N \geq M \) where \( M = \max \{M(k,1), M(k,2), ..., M(k,H)\} \)

\[
\left| \sum_{n=1}^{N} e^{2\pi i < n^2 \gamma >} \right|^2 = \left| \sum_{n=1}^{N} e^{2\pi i k n^2 \gamma} \right|^2 \\
\leq \frac{1}{N^2} \frac{2N}{H} \sum_{h=0}^{H} \left| \sum_{n=1}^{N-h} e^{2\pi i k ((n+h)^2 - n^2) \gamma} \right| \\
= 2 \frac{1}{NH} \sum_{h=0}^{H} \left| \sum_{n=1}^{N-h} e^{2\pi i k (2nh+h^2) \gamma} \right| \\
\leq 2 \frac{1}{NH} \sum_{h=0}^{H} \left| \sum_{n=1}^{N-h} e^{2\pi i k 2nh \gamma} \right| \\
= 2 \frac{1}{NH} (N + \sum_{h=1}^{H} \left| \sum_{n=1}^{N-h} e^{2\pi i k 2nh \gamma} \right| ) \\
\text{the sum is splitted into two part: } h = 0 \text{ and } h \in [1,H] \\
\leq 2 \left( \frac{1}{H} + \frac{1}{H} \epsilon \right) \\
\leq 4 \epsilon
\]

Hence, \( < n^2 \gamma > \) is equidistributed in \([0, 1)\).

(c) Show if \( \{\xi_n\} \) is a sequence of real number so that for all positive \( h \), the difference \( < \xi_{n+h} - \xi_n > \) is equidistributed in \([0, 1)\), then \( < \xi_n > \) is equidistributed in \([0, 1)\).

By Weyl’s criteria, it suffices to show that for any \( k \neq 0 \), \( \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i < \xi_n a > k} \to 0 \) as \( N \to \infty \)

First, for any \( \epsilon > 0 \), there exists \( H > 0 \) s.t. \( \frac{1}{H} \leq \epsilon \)

As \( < \xi_{n+h} - \xi_n > \) is equidistributed for any positive integer \( h \), for any \( k > 0 \) there exists \( M(k,h) > 0 \) s.t.

\[
\left| \sum_{n=1}^{N-H} e^{2\pi i k (\xi_{n+h} - \xi_n)} \right| < \epsilon \text{ for } N \geq M(k,h)
\]
Thus, for any $\epsilon > 0$, for $N \geq M$ where $M = \max\{M_{(k,1)}, M_{(k,2)}, \ldots, M_{(k,H)}\}$

$$\left| \sum_{n=1}^{N} \frac{e^{2\pi i k < \xi_n >}}{N} \right|^2 = \left| \sum_{n=1}^{N} \frac{e^{2\pi i k \xi_n}}{N} \right|^2 \leq \frac{1}{N^2 2^H} \sum_{h=0}^{H} \sum_{n=1}^{N} \left| e^{2\pi i k (\xi_{n+h} - \xi_n)} \right| \leq 2 \left( \frac{1}{H} + \frac{1}{H} \epsilon \right)$$

the sum is splitted into two part: $h = 0$ and $h \in [1, H]$

$$\leq \frac{2}{NH} \left( N + \sum_{h=1}^{H} \sum_{n=1}^{N-h} \left| e^{2\pi i k (\xi_{n+h} - \xi_n)} \right| \right) \leq 4\epsilon$$

Hence, $< \xi_n >$ is equidistributed in $[0, 1)$.

(d) Suppose that $P(x) = c_n x^n + \ldots + c_0$ is a polynomial with real coefficients, where at least one of $c_1, \ldots, c_n$ is irrational. Then the sequence $< P(n) >$ is equidistributed in $[0, 1)$

Claim the following first.

If $P(x) = Q(x) + c_1 x + c_0$ where $c_1$ is irrational and $Q(x) = c_n x^n + \ldots + c_2 x^2$, then $< P(n) >$ is equidistributed in $[0, 1)$

By Weyl’s criteria, it suffice to show that for any $m \neq 0$, $\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i < P(n) > m} \to 0$ as $N \to \infty$
\[
\left| \sum_{n=1}^{N} e^{2\pi miP(n)} \right| = \frac{1}{N} \left| \sum_{n=1}^{N} e^{2\pi miQ(n)} e^{2\pi mic_1 n} e^{2\pi mic_0} \right|
\leq \frac{1}{N} \left| \sum_{n=1}^{N} e^{2\pi miQ(n)} e^{2\pi mic_1 n} \right|
\leq \frac{1}{N} \left| \sum_{n=1}^{N} e^{2\pi miQ(n)} e^{2\pi mic_1 (Bk+d)} \right|
\]

where \( B \) is LCM of the denominator of \( c_0, c_2, \ldots, c_n \)
then every \( n = Bk + d \) for some \( d \in [0, B-1] \), \( k \in [0, \left\lfloor \frac{N}{B} \right\rfloor] \)
it is a division algorithm
\[
\leq \frac{1}{N} \left| \sum_{d=0}^{B-1} \sum_{k=0}^{\left\lfloor \frac{N}{B} \right\rfloor} e^{2\pi miQ(nBk+d)} e^{2\pi mic_1 (Bk+d)} \right|
\leq \frac{1}{N} \left| \sum_{d=0}^{B-1} e^{2\pi miQ(nB+\frac{N}{B}+d)} e^{2\pi mic_1 (B+\frac{N}{B}+d)} \right|
\leq \frac{1}{N} \left| \sum_{d=0}^{B-1} e^{2\pi miQ(d)} e^{2\pi mic_1 (B+d)} \right| + 2B
\]

the sum is splitted into \( k = 0, k = \left\lfloor \frac{N}{B} \right\rfloor \) and the rest.
\[
= \frac{1}{N} \left| \sum_{d=0}^{B-1} \sum_{k=0}^{\left\lfloor \frac{N}{B} \right\rfloor} e^{2\pi miQ(nBk+d)} e^{2\pi mic_1 (Bk+d)} \right| + 2B
\leq \frac{1}{N} \left| \sum_{d=0}^{B-1} e^{2\pi miQ(d)} e^{2\pi mic_1 (B+d)} \right| + 2B
\]
since \( Q(n) \times B \) is a polynomial with integer coefficient,
and \( |e^{2\pi miQ(n)\times B}| = 1 \)
\[
\leq \left| \sum_{d=0}^{B-1} e^{2\pi miQ(d)} \sum_{k=1}^{\left\lfloor \frac{N}{B} \right\rfloor-1} e^{2\pi mic_1 (Bk+d)} \right| + |2B|
\leq \left| \sum_{d=0}^{B-1} e^{2\pi mic_1 (d)} \sum_{k=1}^{\left\lfloor \frac{N}{B} \right\rfloor-1} e^{2\pi mic_1 (Bk)} \right| + |2B|
\leq \left( \sum_{d=0}^{B-1} \left| e^{2\pi mic_1 (d)} \right| \sum_{k=1}^{\left\lfloor \frac{N}{B} \right\rfloor-1} e^{2\pi mic_1 (Bk)} \right) + |2B|
\leq \left( \sum_{d=0}^{B-1} \left| e^{2\pi mic_1 (d)} \right| \sum_{k=1}^{\left\lfloor \frac{N}{B} \right\rfloor-1} e^{2\pi mic_1 (Bk)} \right) + |2B|
\]
as \( Bc_1 \) is irrational, \( Bc_1 \) is equidistributed, i.e. for there exists \( M > 0 \) s.t.
for \( N > M \), we have \( |\frac{1}{N} e^{2\pi mic_1 (Bk)}| < \frac{\epsilon}{B} \)
Indeed we choose larger \( M \) s.t. \( \frac{2B}{M} < \epsilon \).
Hence, we have for any \( m \neq 0 \), for any \( \epsilon > 0 \), for \( N > M \)

\[
\left| \sum_{n=1}^{N} \frac{e^{2\pi i m n P(n)}}{N} \right| \leq \left( \sum_{d=0}^{B-1} \sum_{k=1}^{[N/B]-1} |e^{2\pi i mk(c_1 B)}| + |2B| \right) \\
\leq \epsilon + \frac{2B}{N} \\
\leq 2\epsilon
\]

Hence, by Weyl’s criteria, \( < P(x) = Q(x) + c_1(x) + c_0 > \) is equidistributed.

Then, Let \( F(k) \): The polynomial sequence \( < P(n) > \) whose the highest degree of \( P \) with irrational coefficient is \( k \) is equidistributed.

(i.e. \( P(x) = x^3 + \sqrt{2}x^2 + \pi x \), then the highest degree of \( P \) with irrational coefficient is 2)

By the above, \( k = 1 \) is true.

Assume \( k = m \) is true,

for \( k = m + 1 \), let \( p(n) \) be the polynomial sequence whose the highest degree with irrational coefficient is \( m + 1 \).

Then for any positive integer \( h \), \( p(n + h) - p(n) \) is a polynomial whose the highest degree with irrational coefficient is \( m \).

As \( k = m \) is true, \( < p(n + h) - p(n) > \) is equidistributed for every positive integer \( h \).

By 2c, \( < p(n) > \) is equidistributed.

Thus, \( F(k + 1) \) is true

By M.I., \( F(k) \) is true for all positive integer \( k \).

If there are any mistakes in the assignment, please contact me through email.