Review Problems for Final Exam.

1. Let

$$\phi(x) = \begin{cases} 
-x + a, & 0 < x < 1 \\
-1 + x, & 1 < x < 2
\end{cases}$$

(a) Determine a so that the Fourier series of $\phi$ is pointwise convergent to $\phi$ in $(0, 2)$. Find the corresponding series.

(b) Find $\phi_{\text{odd}}$, the odd extension of $\phi$. Determine a so that the full Fourier series of $\phi_{\text{odd}}$ is convergent to $\phi_{\text{odd}}$ at $0$.

Proof. For the pointwise convergence of the Fourier series of $\phi$, $\phi$ have to be continuous on $(0, 2)$ and have right-hand and left-hand derivative on $(0, 2)$. We have to choose a $a$ s.t. $\phi(x)$ is continuous at $x = 1$, and we will such $\phi(x)$ is piecewise differentiable on $(0, 1)$.

i.e. $-1 + a = -1 + 1$.

$$a = 1$$
The corresponding series,
\[ A_0 = -\int_0^2 \phi(x) \, dx \]
\[ = -\int_0^1 -x+1 \, dx + \int_1^2 -1+x \, dx \]
\[ = -\frac{x^2}{2} \bigg|_0^1 + \frac{x^2}{2} \bigg|_1^2 \]
\[ = -\frac{1}{2} + 2 - \frac{1}{2} \]
\[ = 1 \]

\[ A_m = \int_0^2 \phi(x) \cos mx \, dx \]
\[ = \int_0^1 (-x+1) \cos mx \, dx + \int_1^2 (-1+x) \cos mx \, dx \]
\[ = \int_0^1 (1-x) \cos mx \, dx + \int_0^1 x \cos(mx + \pi m) \, dx \quad \text{(substitution } x = x-1) \]
\[ = \begin{cases} 
\int_0^1 \cos(mx) \, dx & \text{if } m \text{ is even} \\
\int_0^1 (1-2x) \cos(mx) \, dx & \text{if } m \text{ is odd} 
\end{cases} \]
\[ = \begin{cases} 
0 & \text{if } m \text{ is even} \\
\frac{4}{(mn)^2} & \text{if } m \text{ is odd}. 
\end{cases} \]
\[ B_m = \int_0^2 \phi(x) \sin m\pi x \, dx \]

\[ = \int_0^1 (-x+1) \sin m\pi x \, dx + \int_1^2 (-1+x) \sin (m\pi x) \, dx \]

\[ = \int_0^1 (1-x) \sin m\pi x \, dx + \int_0^1 x \sin (m\pi x + m\pi) \, dx \]

\[ = \begin{cases} 
\int_0^1 \sin (m\pi x) \, dx & \text{if } m \text{ is even} \\
\int_0^1 (1-2x) \sin (m\pi x) \, dx & \text{if } m \text{ is odd}.
\end{cases} \]

\[ = \begin{cases} 
0 & \text{if } m \text{ is even} \\
0 & \text{if } m \text{ is odd}.
\end{cases} \]

\[ \therefore \text{ The corresponding full Fourier Series is} \]

\[ \phi(x) = 1 + \sum_{m \text{ odd}} \frac{4}{(mn)^2} \cos (m\pi x) \]

\[ \text{ } \]
b) $\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{if } 0 < x < 2 \\ -\phi(-x) & \text{if } -2 < x < 0 \end{cases}$

For the pointwise convergence of the Fourier series of $\phi$ at $x=0$

$\phi_{\text{odd}}$ have to be continuous at $x=0$ and $\phi_{\text{odd}}$ has right hand derivative and the left hand derivative at $x=0$,

$\therefore \phi_{\text{odd}}(0) = 0 \Rightarrow -0 + a = 0 \Rightarrow a = 0$

$\therefore \phi_{\text{odd}}$ has right hand and left-hand derivative at $x=0$ and $\phi_{\text{odd}}$ is continuous at $x=0$

$\therefore$ The full Fourier series of $\phi$ pointwise converge to $\phi$ at $x=0$.

ref: See the theorem 4.10 on your textbook 1.129
Q2.

Solve the following equations.

\[ U_t = k U_{xx} \quad \text{for} \quad x \in (-\pi, \pi) \]

\[ U(-\pi, t) = U(\pi, t) \quad \text{(i)} \]

\[ U_x(-\pi, t) = U_x(\pi, t) \quad \text{(ii)} \]

\[ U(x, 0) = \sin(2x) - \cos(6x) \]

\[ Pf: \quad \text{Let} \quad U = X(x) T(t) \]

\[ \therefore \quad U_t = k X_{xx} \quad \text{becomes} \]

\[ \frac{T'}{k T} = \frac{X''}{X} = -\lambda \]

for \( \lambda = \beta^2 > 0 \)

\[ X'' = -\beta^2 X \]

\[ X = A \cos \beta x + B \sin \beta x \]

by (i), \( A \cos \beta \pi + B \sin \beta \pi = A \cos \beta \pi - B \sin \beta \pi \)

\[ \Rightarrow \quad \sin \beta \pi = 0 \]

\[ \beta = n \]

\[ \therefore \]
by (iii), \(-A\beta \sin \beta \pi + B\beta \cos \beta \pi = -A\beta \sin \beta \pi + B\beta \cos \beta \pi\)

\[\Rightarrow \quad \sin \beta \pi = 0\]

\[\Rightarrow \quad \beta = n\]

For \(\lambda = 0\)

\[
\mathbf{Z} = a\mathbf{x} + b
\]

by (i) \(a(-\pi) + b = a(\pi) + b \Rightarrow a = 0\), \(b\) can be any number.

For \(\lambda = -\beta^2 < 0\)

\[
\mathbf{Z}'' = \beta^2 \mathbf{Z}
\]

\[\Rightarrow \quad \mathbf{Z} = A e^{-\beta x} + B e^{\beta x}\]

by (i), \(A e^{-\beta x} + B e^{\beta x} = A e^{\beta x} + B e^{-\beta x}\)

by (ii), \(-\beta A e^{-\beta x} + B \beta e^{\beta x} = -A \beta e^{\beta x} + B \beta e^{-\beta x}\)

\[A = B = 0\] (I did similar thing in the past.

like review mid-term 2)

\[\text{reject.}\]
Eigenvalues are \( n^2 \) for \( n \geq 0 \), \( n \in \mathbb{N} \)

Eigenfunctions are \( 1, A_n \cos nx + B_n \sin nx \)

\[
T(t) = \begin{cases} 
A_0 & \text{for } \lambda = 0 \\
\sum_{n=1}^{\infty} e^{-n^2 k t} & \text{for } \lambda = n^2 
\end{cases}
\]

\[
U(x, t) = A_0 + \sum_{n=1}^{\infty} e^{-n^2 k t} (A_n \cos nx + B_n \sin nx)
\]

At \( t = 0 \),

\[
\sin(2x) - \cos(6x) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx.
\]

By the uniqueness of the Fourier Series,
we have \( A_n = \begin{cases} 
\frac{1}{2} & \text{if } n = 6 \\
0 & \text{otherwise}
\end{cases} \)

\( B_n = \begin{cases} 
1 & \text{if } n = 2 \\
0 & \text{otherwise}
\end{cases} \)

\[
U(x, t) = e^{-4k t} \sin 2x - e^{-36k t} \cos 6x
\]

\( \& \)
Q3

Solve $u_{xx} + u_{yy} = \frac{u}{x^2 + y^2}$ in the annulus $1 \leq r \leq 2$.

with $u$ vanishes on the inner boundary and $u = 3$ on the outer boundary.

pf: Since $u = \text{const}$ is a constant function on the inner and outer boundary respectively.

We can guess $u$ is rotational invariance.

Once you can find a solution, by the uniqueness, we can say it is the only solution.

\[ u_{xx} + u_{yy} = \frac{u}{x^2 + y^2} \]

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u = \frac{u}{r^2}
\]

\[ u \text{ is rotational invariance.} \]

Take $u = rV$, we have

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) (rV) = \frac{V}{r}
\]
\[ \frac{\partial^2 V}{\partial r^2} + \frac{3}{r} \frac{\partial V}{\partial r} = 0 \]

\[ \frac{\partial V}{\partial r} = C e^{-3br} \quad \text{where } C \text{ is a constant.} \]

\[ = \frac{C}{r^3} \]

\[ V(r) = \int_1^r \frac{C}{s^3} \, ds + V(1) \]

\[ \frac{u(r)}{r} = \int_1^r \frac{C}{s^3} \, ds + \frac{u(1)}{4} \]

\[ u(r) = r \int_1^r \frac{C}{s^3} \, ds + u(1) \]

\[ = r \int_1^r \frac{C}{s^3} \, ds \]

At \( r = 2 \), \( u(r) = 3 \)

\[ 3 = 2 \int_1^2 \frac{C}{s^2} \, ds = \frac{3C}{4} \]

\[ C = 4 \]

\[ u(r) = 4r \int_1^r \frac{C}{s^3} \, ds \]
Q4.

Show that the Robin condition given by

\[
\begin{cases}
    z'(0) - \alpha z(0) = 0 \\
    z'(l) + \alpha z(l) = 0
\end{cases}
\]

doesn't have negative eigenvalue if \( \alpha > 0 \).

Pf.: By the theorem 3 on your textbook p.122, it suffices to show if \( f(x) \) and \( g(x) \) satisfy the BCs (*), then

(i) \( f'g - g'f \bigg|_{x=0}^{x=l} = 0 \) (symmetric boundary)

(ii) \( f(x)f'(x) \bigg|_{x=0}^{x=l} \leq 0 \)

To show (i)

\[
f'g - g'f \bigg|_{x=0}^{x=l} = f'(0)g(0) - g'(0)f(0) - f'(l)g(l) - g'(l)f(l)
\]
\[- a f(l) g(l) - (-a g(l) f(l)) \]
\[- a f(0) g(0) + a g(0) f(0) \]
\[= -0. \]

To check (ii),
\[f(\alpha) f'(\alpha) \bigg|_{x=\ell} = f(l) f'(l) - f(0) f'(0) \]
\[= - a (f'(l))^2 - a(f'(0))^2 \]
\[\leq 0 \quad \text{since} \quad a \geq 0 \]
Q5.
Solve the following equation.

\[ u_t = ku_{xx} \quad \text{for} \quad x \in (-\infty, \infty), \ t \in (0, \infty) \]

\[ u(x, 0) = x \]

pf: By the equation 2.4.8 on the textbook p. 49, we have

\[ u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} \, dy \]

Let \( z = \frac{y-x}{\sqrt{4kt}} \), so we have

\[ u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-z^2} \, dz \sqrt{4kt} \, dz + \frac{t}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-z^2} \, dz \]

\[ = 0 - \frac{t}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} \, dz \quad \text{odd function w.r.t.} \ z \]

\[ = x \]