Summary about Full Fourier Series

Convergence of the Fourier Series of $f$ on $[a, b]$

<table>
<thead>
<tr>
<th>Condition</th>
<th>$L^2$-convergence</th>
<th>Pointwise Convergence</th>
<th>Uniformly Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $\int_a^b</td>
<td>f</td>
<td>^2 , dx &lt; \infty$</td>
<td>✓</td>
</tr>
</tbody>
</table>

If $f(x)$ is continuous and $f'(x)$ is piecewise continuous on $[a, b]$

|                        | ✓                  | ✓                     |                       |

Remark:
- If pointwise converges to $f$ on $[a, b]$

If $f(x)$ and $f'(x)$ are piecewise continuous on $[a, b]$

|                        | ✓                  | ✓                     | ✓                     |

Remark:
- If pointwise converges but the limit may not be $f(x)$ for some $x \in [a, b]$

If $f(x)$, $f'(x)$, $f''(x)$ are continuous on $[a, b]$

|                        | ✓                  | ✓                     | ✓                     |

Remark: continuity at the end-pt. mean when you extend the function periodically, the function is continuous at the pt. $a$ & $b$.

It is what I talked in a session.
Uniformly Convergence Theorem for Fourier Sine Series
If \( f, f', f'' \) are continuous on \([a, b]\)
and \( f(a) = f(b) = 0 \)
then its Fourier Sine Series is uniformly convergent
to \( f \) on \([a, b]\).

Uniformly Convergence Theorem for Fourier Cosine Series
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The Uniqueness Theorem of a Full-Fourier Series
If \( f \) is \( \text{continuous} \) on \([a, b]\) and suppose
\( f \) can be written down as two different full Fourier Series, i.e.
\[
\begin{align*}
f(x) & \equiv A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{b-a} + B_n \sin \frac{n\pi x}{b-a} \\
& \equiv A'_0 + \sum_{n=1}^{\infty} A'_n \cos \frac{n\pi x}{b-a} + B'_n \sin \frac{n\pi x}{b-a}
\end{align*}
\]
then \( A_0 = A'_0 \), \( A_n = A'_n \) and \( B_n = B'_n \).