Week 2.
I will do P.9 #2, #3, #5, #6, #9
& P.19 #1

P.9 Q2
Solve the equation $3u_y + u_{xy} = 0$

Pf: Let $v = u_y$.

$3u_y + u_{xy} = 0$ becomes

$3v + v_x = 0$

$v = C(y)e^{3x}$ \(\text{ (it is obtained by solving the above ODE) }\)

where $C(y)$ is a function of $y$.

$u_y = C(y)e^{3x}$

$u = e^{3x}\int_0^y C(t)\,dt + D$ \text{ where $D$ is a constant.}

$u = e^{3x}g(y) + D$ \text{ where $g = \int_0^y C(t)\,dt$ is a $C'$ function.}
Q3 Solve the equation \((1 + x^2)u_x + uy = 0\). Sketch some of the characteristic curves.

**Pf:** Let \((x(s), y(s))\) be a characteristic curve. We have
\[
\begin{align*}
\dot{x}(s) &= 1 + x^2 \\
y(s) &= 1
\end{align*}
\]

\[
\therefore \quad \frac{dy}{dx} = \frac{1}{1 + x^2}
\]

\[
\therefore \quad \frac{dx}{1 + x^2} = dy
\]

\[
\therefore \quad x = \tan (y + c)
\]

and \(c = \tan^{-1}x - y\).

\[
\therefore \quad u(x, y) = f(c) = f(\tan^{-1}x - y) \quad \text{for some } c'
\]

And the characteristic curve will look like:

\[
\begin{align*}
c &= 0 \\
c &= \tan^{-1}x - y
\end{align*}
\]
Q5. Solve the equation \( xu_x + y u_y = 0 \)

pf: Let \((x(s), \ y(s))\) be a characteristic curve.

\[
\begin{align*}
\dot{x}(s) &= x \\
\dot{y}(s) &= y
\end{align*}
\]

\[
\frac{dy}{dx} = \frac{y}{x}.
\]

\( y = Cx \)

\[
\therefore u(x, y) = f(C) = f\left(\frac{y}{x}\right)
\]

where \( f \) is a \( C^1 \) function.
28. Solve \( au_x + bu_y + cu = 0 \)

**Proof:** Let \((x(s), y(s))\) be a characteristic curve.
\[
\begin{align*}
\dot{x}(s) &= \alpha \\
\dot{y}(s) &= \beta \\
\frac{dy}{dx} &= \frac{\beta}{\alpha} \\
y &= \frac{\beta}{\alpha} x + d
\end{align*}
\]
We note that \( u \) can be expressed by \( x \)
\[
\begin{align*}
y(x) &= \frac{\beta}{\alpha} x + d \\
\frac{d}{dx} u(x, y(x)) &= -\frac{c}{\alpha} u \\
u(0, d) &= f(d)
\end{align*}
\]
\[=\]
\[
u(x, y) = D e^{-\frac{c}{\alpha} x} \\
= f(d) e^{-\frac{c}{\alpha} x} \\
= f(y - \frac{\beta}{\alpha} x) e^{-\frac{c}{\alpha} x}
\]

**Note:** Although the answer in your book is
\[f(ay - bx) e^{-\frac{c}{\alpha} x(x + by)}, \quad \exists \tilde{f}(ay - bx) \text{ s.t.} \tilde{f}(ay - bx) e^{-\frac{c}{\alpha} x} = f(ay - bx) e^{-\frac{c}{\alpha} (ax + by)}\]
Q9. Solve the equation \( u_x + u_y = 1 \).

Let \((x(s), y(s))\) be a characteristic curve.

\[
\begin{align*}
\dot{x}(s) &= 1 \\
\dot{y}(s) &= 1 \\
\frac{dy}{dx} &= 1
\end{align*}
\]

\[
y = x + C
\]

Following a similar idea in Q8, we have

\[
\begin{align*}
\int \frac{d}{dx} u(x, y(x)) &= 1 \\
\int u(0, c) &= f(c) \\
\Rightarrow u(x, y) &= c + x \\
&= f(c) + x \\
&= f(y-x) + x
\end{align*}
\]