Q1 Use the energy conservation of the wave equation to prove that the only solution with \( \phi \equiv 0 \) and \( \psi \equiv 0 \) is \( u \equiv 0 \).

(Hint: Use the first vanishing theorem in A1.)

**Proof:**

\[
E(t) = \frac{1}{2} \int_{-\infty}^{\infty} (\rho u_t^2) + (T u_x^2) \, dx
\]

On textbook R40, \( \frac{dE}{dt} = 0 \)

\[
E(t) = E(0) = \frac{1}{2} \int_{-\infty}^{\infty} \rho \psi^2(x) + T (\phi_x)^2 \, dx
\]

\[= 0\]

Also, \( \rho u_t^2 + T u_x^2 \geq 0 \) on \( \mathbb{R} \)

\[
\therefore \quad u_t = u_x = 0 \quad \text{for} \quad t > 0, \ x \in \mathbb{R}
\]

\[
\therefore \quad u \equiv 0 \quad \text{on} \quad \mathbb{R}^+ \times \mathbb{R}
\]
#2 For a solution \( u(x,t) \) of the wave equation with 
\( p = T = c = 1 \), the energy density is defined as 
\[
e = \frac{1}{2} (u_t^2 + u_x^2)
\]
and the momentum density as 
\[
p = u_t u_x.
\]

(a) Show that 
\[
\frac{\partial e}{\partial t} = \frac{\partial p}{\partial x} \quad \text{and} \quad \frac{\partial p}{\partial t} = \frac{\partial e}{\partial x}
\]
(b) Show that both \( e(x,t) \) and \( p(x,t) \) also satisfy the wave equation.

\[
\begin{align*}
\frac{\partial e}{\partial t} &= u_t u_{tt} + u_x u_{xt}, \quad \text{or} \quad \frac{\partial p}{\partial x} = u_t u_x + u_t u_{xx} \\
&= u_x u_x + u_t u_{tt} \quad \text{(\textit{sol of the wave \( e \))}}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial p}{\partial t} &= u_{tt} u_x + u_t u_{xt} \quad \text{and} \quad \frac{\partial e}{\partial x} = u_t u_{tx} + u_x u_{xx} \\
&= u_t u_{tx} + u_x u_{tt}
\end{align*}
\]

\[
\therefore \quad \frac{\partial e}{\partial t} = \frac{\partial p}{\partial x}
\]

(b) \( e_{tt} = (p_x)_x = (p_t)_x = (p_x)_t = (e_x)_x = e_{xx} \) and 
\[
(p_{tt}) = (p_t)_t = (e_x)_t = (e_t)_x = (p_x)_x = p_{xx}
\]
25. For the damped string, equation (1.3.3), show that the energy decreases.

\[ \text{Pf: The wave equation of the damped string:} \]
\[ u_{tt} - c^2 u_{xx} + ru_t = 0 \quad \text{where } r > 0. \]

\[ \therefore \quad E(t) = \frac{1}{2} \int_{-\infty}^{\infty} (u_t^2 + c^2 u_x^2) \, dx \]

\[ \frac{dE}{dt} = \frac{1}{2} \int_{-\infty}^{\infty} 2u_t u_{tt} + 2c^2 u_x u_{xt} \, dx \]

\[ = \int_{-\infty}^{\infty} u_t (c^2 u_{xx} - ru_t) + c^2 u_x u_{xt} \, dx \]

\[ = \int_{-\infty}^{\infty} u_t c^2 u_{xx} \, dx - \int_{-\infty}^{\infty} ru_t^2 \, dx + c^2 u_x u_t(x = \infty) \]

\[ = -\int_{-\infty}^{\infty} ru_t^2 \, dx \]

\[ \leq 0 \]

as just like your textbook p.48, we assume \( \phi(x) \) and \( \psi(x) \) vanish outside an interval \( \{ |x| \leq R \} \).

\[ \therefore \quad \text{The solution } u \quad \text{will have the following property:} \]
\[ \text{for any fixed } t_0, \]
\[ u(\pm x, t) = 0 \quad \text{for } R \text{ large,} \]
\[ \quad \text{for } |x| > R \]
Q1 Consider the solution \(1 - x^2 - 2kt\) of the diffusion equation. Find the location of its maximum and its minimum in the closed rectangle.
\[0 \leq x \leq 1, \quad 0 \leq t \leq T^2\]

**Proof:** Let \(E = \{0 \leq x \leq 1, \quad 0 \leq t \leq T^2\}\),

\[\partial E = \{0 \leq x \leq 1, \quad t = 0\} \cup \{x = 0, \quad 0 \leq t \leq T\} \cup \{x = 1, \quad 0 \leq t \leq T\}\]

Let \(u(x, t) = 1 - x^2 - 2kt\.

As \(u\) is a solution of the diffusion equation, by the max principle,

\[
\max_{(x, t) \in E} u(x, t) = \max_{(x, t) \in \partial E} u(x, t)
\]

\[
\min_{(x, t) \in E} u(x, t) = \min_{(x, t) \in \partial E} u(x, t)
\]

Since \(t > 0\), \(-x^2 - 2kt < 0\) for \(\forall (x, t) \in \partial E\) except \(x = t = 0\)

\[
\max_{(x, t) \in \partial E} u(x, t) = u(0, 0) = 1
\]

Also, on \(\partial E\), \(-x^2 - 2kt \geq -1^2 - 2kT\)

\[
\min_{(x, t) \in \partial E} u(x, t) = u(1, T) = -2kT.
\]

\((1, T)\) is min location while \((0, 0)\) is the max location.
Q2
Consider a solution of the diffusion equation \( u_t = u_{xx} \) in 
\( \{0 \leq x < l, \ 0 \leq t < \infty\} \).

(a) Let \( M(T) = \max_{\{0 \leq x < l, \ 0 \leq t \leq T\}} u(x, t) \) in the closed rectangle \( \{0 \leq x < l, \ 0 \leq t \leq T\} \). Does \( M(T) \) increase or decrease as a function of \( T \)?

(b) Let \( m(T) = \min_{\{0 \leq x < l, \ 0 \leq t \leq T\}} u(x, t) \) in the closed rectangle \( \{0 \leq x < l, \ 0 \leq t \leq T\} \). Does \( M(T) \) increase or decrease as a function of \( T \)?

Proof:
(a) \( M(T) \) increases as \( T \) increases.

Let \( T_1 < T_2 \).

Then \( M(T_1) = \max_{\{0 \leq x < l, \ 0 \leq t \leq T_1\}} u(x, t) \)

\[ \leq \max_{\{0 \leq x < l, \ 0 \leq t \leq T_2\}} u(x, t) \]

\[ = M(T_2) \]

(b) By the similar argument to the part (a), we have \( m(T) \) decreases as \( T \) decreases.
Q4.

Consider the diffusion equation \( u_t = u_{xx} \) in 
\( 0 < x < 1, \ 0 < t < \infty \), with \( u(0, t) = u(1, t) = 0 \)
and \( u(x, 0) = 4x (1 - x) \).

(a) Show that \( 0 < u(x, t) < 1 \) for all \( t > 0 \) and \( 0 < x < 1 \).
(b) Show that \( u(x, t) = u(1-x, t) \) for all \( t > 0 \) and 
\( 0 < x < 1 \).
(c) Use the energy method to show that \( \int_0^1 u^2 \, dx \)
is a strictly decreasing function of \( t \).

\[ \text{Pf:} \ (a) \ \text{Note the set } \{ 0 < x < 1, \ 0 < t < \infty \} \text{ has } \]
\( t \) runs from 0 to \( \infty \). So \( u_t \) is 
not bounded and we can't use the 
\underline{max. principle theorem} on your textbook 
\[ \text{P. 42.} \]
Instead, we use a strong max. principle 
of a diffusion equation \( u_t = \kappa u_{xx} \).
"Strong max. principle"

Given \( U_t = U_{xx} \) in \( \{0 < x < 1, \ 0 < t \leq T\} \), (in this case \( T \) can be \( t = \infty \)), then we have

if there exists \((x_0, t_0) \in \{0 < x < 1, \ 0 < t \leq T\} \cup \{t = T\}\)

s.t. \( U(x_0, t_0) = \max_{\{0 \leq x \leq 1, \ 0 \leq t \leq T\}} U(x, t) \)

then \( U(x, t) \equiv U(x_0, t_0) \) in \( \{0 \leq x \leq 1, \ 0 \leq t \leq T\} \)

i. By the strong max. principle,

we have

\[
\min_{\{x=0, \ t \geq 0\}} U(x, t) < U(x, t) < \max_{\{x=1, \ t \geq 0\}} U(x, t)
\]

\[
\min_{\{0 < x < 1, \ t = 0\}} U(x, t) < U(x, t) < \max_{\{0 < x < 1, \ t = 0\}} U(x, t)
\]

\[
0 < U(x, t) < 1
\]
(b) Let \( V(x, t) = U(1-x, t) \)

as \[ \begin{cases} 
V_t = V_{xx} \\
V(0, t) = V(1, t) = 0 \text{ and } V(x, 0) = 4x(1-x)
\end{cases} \]

at the same time,

\[ \begin{cases} 
U_{xx} = U_t \\
U(0, t) = U(1, t) = 0 \text{ and } U(x, 0) = 4x(1-x)
\end{cases} \]

\( V \) & \( U \) are the soln of the diffusion equation with the same initial conditions.

By the \( uniqueness \) of the soln of the diffusion equation, we have

\[ V(x, t) = U(x, t) \]

\( U(1-x, t) = U(x, t) \)
\[
\frac{d}{dt} \int_0^1 u^2 \, dx = \int_0^1 2u \, u_t \, dx = \int_0^1 2u \, u_{xx} \, dx.
\]

\[
\therefore u_t = u_{xx}
\]

\[
= -2 \int_0^1 u_x^2 \, dx 
\]

\[
< 0
\]

\[
(e) \text{ for fixed } t_0, \quad u(0, t_0) = u(1, t_0) = 0
\]

\[
\text{but } 0 < u(x, t) \text{ for } 0 < x < 1
\]

\[
\Rightarrow u_x(x, t_0) \text{ is not identically zero}
\]

\[
\Rightarrow -\int_0^1 u_x^2 \, dx < 0.
\]
Q5
The purpose of this exercise is to show that the maximum principle is not true for the equation \( u_t = xu_{xx} \) which has a variable coefficient.

(a) Verify that \( u = -2x^2 - x^2 \) is a solution. Find the location of its maximum in the closed rectangle \( \{ -2 \leq x \leq 2, 0 \leq t \leq 1 \} \).

(b) Where precisely does our proof of the maximum principle break down for this equation.

\[ Pf : (a) \quad u_t = -2x \]
\[ u_{xx} = -2 \]
\[ \therefore \quad u_t = xu_{xx}. \]

The location of the max. in the closed rectangle is \( u(0, 0) = 0 \) while the location of the max. in the boundary of the rectangle is \( u(-2, 1) = 0 \) it fails in the strong max. principle.
\[ U(x,t) = t^2 - (t+x)^2 \]

: The location of the max. in the closed rectangle is 
\[ U(-1,1) = 1 \]

However, the location of the max

the max. of \( U \) on \( \{ x=0 \}, 0 \leq t \leq 15 \) \( \{ x=-2 \}, 0 \leq t \leq 15 \) 
\[ \{ x \geq t=0 \}, -2 \leq x \leq 2 \] is less than 1.

: The max. principle fails in this case.

(b) The reason is as follows.

at \((-1,1)\),
\[ U_t \geq 0, \quad U_{xx} = -2 \leq 0, \]
it does satisfy the rule of 
\[ U_{xx} \leq 0 \] calculus if the interior pt. is max.

also, \( U_{xx} \cdot U = U_t \) still holds at \((-1,1)\).

So, we cannot make a contradiction argument at this step where the proof in this book.

\[ U_t \geq 0 \]

it is used in
Q6. Prove the comparison principle for the diffusion equation.

If \( u \) and \( v \) are two solutions and if \( u \leq v \) for \( t=0 \), \( x=0 \), and \( x=l \), then \( u \leq v \) for \( 0 \leq t < \infty, \ 0 \leq x \leq l \).

**pf:** Let \( w = v - u \).

\[
W_{xx} = u_{xx} - u_{xx} = V_t - U_t = W_t
\]

And on \( \{ t=0 \}, \ \{ x=0 \}, \ \{ x=l \} \),

\[
W = v - u \geq 0
\]

\[\therefore\] By the max. principle, we have \( u \geq v \) on for \( 0 \leq t < \infty, \ 0 \leq x \leq l \).
Q7.
(a) More generally, if \( u_t - ku_{xx} = f \), \( v_t - kv_{xx} = g \),
    \( f \leq g \) and \( u \leq v \) at \( x = 0, \ x = l \) and \( t = 0 \),
    prove that \( u \leq v \) for \( 0 \leq x \leq l, \ 0 \leq t \leq \infty \).

(b) If \( v_t - v_{xx} \geq \sin x \) for \( 0 \leq x \leq \pi, \ 0 \leq t \leq \infty \),
    and \( f \neq v(0, t) = 0, \ v(\pi, t) = 0 \) and \( \nabla(v(x, 0)) \geq \sin x \),
    use part (a) to show that \( \nabla v(x, t) \geq (1 - e^{-t}) \sin x \).

\( \text{If:} \quad (a) \quad \boxed{\text{Fact: I hope you know how to prove it}} \)

\[ \text{If} \quad u_t - ku_{xx} \leq 0 \quad \text{in} \quad R = [0, l] \times [0, T], \quad \text{then} \]
\[ \max_{R} u(x, t) = \max_{\partial R} u(x, t) \]

\[ \text{then let } \ w = \frac{1}{v} (u - \nabla v) \]
\[ \text{follow the steps in Q6.} \]
\[ \text{part a is done.} \]
(b) Let \( u(x, t) = (1 - e^{-t}) \sin x \).

Then \( u(0, t) = 0 \), \( u(\pi, t) = 0 \), \( u(x, 0) = 0 \)

and \( u_t - u_{xx} = \sin x \).

By part (a), we have

\[ v(x, t) \geq (1 - e^{-t}) \sin x. \]