Q1
Solve the diffusion problem $u_t = k u_{xx}$ in $0 < x < l$, with the mixed boundary conditions $u(0, t) = u_x(l, t) = 0$.

Pf: Let $u(x, t) = T(t) \varphi(x)$

$$\frac{T'}{\frac{d}{dT}} = \frac{\varphi''}{\varphi} = -\lambda$$

Instead of considering four cases of $\lambda$

i.e. $\lambda > 0$, $\lambda < 0$, $\lambda = 0$, $-\lambda$ = complex number.

It suffices for us to consider $\lambda$ being a complex number only because it is the most general case.

If $-\lambda = a$ complex number = $\beta^2$.

Case (i) $\beta \neq 0$

$$\frac{\varphi''}{\varphi} = -\lambda$$

$$\Rightarrow \varphi(x) = Ce^{\beta x} + De^{-\beta x}$$

$$\varphi'(x) = C\beta e^{\beta x} - D\beta e^{-\beta x}$$
\[ x(0) = 0 \Rightarrow C + D = 0 \]
\[ x'(l) = 0 \Rightarrow Ce^{\beta l} = De^{-\beta l} \]
\[ \Rightarrow e^{\beta l} = -e^{-\beta l} \]
\[ \Rightarrow e^{2\beta l} = -1 \]
\[ \text{we can assume } C \neq 0 \text{ & } D \neq 0. \]
\[ \text{because } C + D = 0 \text{ & we don't want a trivial case } C = D = 0. \]
\[ \text{Im} (\beta) = \frac{(n + \frac{1}{2})\pi}{l} , \quad \text{Re} (\beta) = 0 \]
\[ \Rightarrow \lambda = \left[ \frac{(n + \frac{1}{2})\pi}{l} \right]^2 \quad \text{for } n = 0, 1, 2, \ldots \]
\[ \Rightarrow \bar{x}(x) = \sin \left( \frac{(n + \frac{1}{2})\pi}{l} \right) x. \]

\[ \text{case (ii) } \beta = 0 \]
\[ \Rightarrow \bar{x}(x) = Cx + D \]
\[ \therefore \bar{x}(0) = 0 \quad \text{and} \quad \bar{x}'(l) = 0 \Rightarrow C = D = 0 \]
\[ \therefore \beta = 0 \text{ is rejected.} \]

\[ \text{To sum up}, \]
\[ \text{eigenvalues} = \lambda_n = \left[ \frac{(n + \frac{1}{2})\pi}{l} \right]^2 \]
\[ \text{eigenfunctions} = \bar{x}_n = \sin \left( \frac{(n + \frac{1}{2})\pi}{l} \right) x \]
And \[ \frac{T_n'}{kT_n} = \lambda_n \]

\[ \Rightarrow T_n = A_n e^{-k\lambda_n t} \]

\[ u(x, t) = \sum_{n=0}^{\infty} A_n e^{-k\left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2 t} \sin\left(L\frac{(n+\frac{1}{2})\pi}{L} x\right) \]
Q2.
Consider the equation \( \ddot{u} + c^2 \dot{u} = 0 \), for \( 0 < x < l \), with the boundary conditions \( u_x(0, t) = 0 \), \( u(l, t) = 0 \).

(Neumann at the left, Dirichlet at the right)

(a) Show that the eigenfunctions are \( \cos \left[ \left( n + \frac{1}{2} \right) \frac{\pi x}{l} \right] \).

(b) Write the series expansion for a solution \( u(x, t) \).

\[ \text{Pf: I found an excellent proof online to solve (a). Here you go.} \]

(a) Let \( u(x, t) = T(t) \times(x) \)

\[ \Rightarrow \frac{T''}{c^2T} = \frac{\times''}{\times} = -\lambda \]

To consider \( \times'' + \lambda \times = 0 \) for \( 0 < x < l \)

\[ \begin{cases} \\ \times'(0) = 0 \\ \times(l) = 0 \end{cases} \]

Here is the thing, we extend the domain of \( \times(x) \) on \([0, l]\) to \([-l, l]\) by an even reflection.
\[ \tilde{\mathcal{X}}(x) := \begin{cases} \mathcal{X}(x) & \text{if } 0 \leq x < l \\ \mathcal{X}(-x) & \text{if } -l < x < 0 \end{cases} \]

Since \( \tilde{\mathcal{X}}'(0) = 0 \),

\[ \therefore \tilde{\mathcal{X}}(x) \text{ is differentiable at } x = 0. \]

and \( \tilde{\mathcal{X}}(0) = 0 \)

Since \( \tilde{\mathcal{X}}'(x) = \begin{cases} \mathcal{X}'(x) & \text{if } 0 \leq x < l \\ -\mathcal{X}'(-x) & \text{if } -l < x < 0 \end{cases} \)

You can check \( \tilde{\mathcal{X}}''(x) \) exists for \( -l < x < l \)

Moreover, we have

\[ \tilde{\mathcal{X}}'' + \lambda \tilde{\mathcal{X}} = 0 \text{ for } -l < x < l \]

\[ \tilde{\mathcal{X}}(l) = \tilde{\mathcal{X}}(l) = 0 \]

Now, we change \( \tilde{\mathcal{X}} \) a little bit by letting

\[ \tilde{\mathcal{Y}}(x) = \tilde{\mathcal{X}}(x - l) \text{ for } 0 < x < 2l \]
\[
\begin{aligned}
\begin{cases}
\dddot{\gamma} + \lambda \dot{\gamma} = 0 & \text{for } 0 < x < 2l \\
\dot{\gamma}(0) = \dot{\gamma}(2l) = 0
\end{cases}
\end{aligned}
\]

Now, it becomes the case 2 in Ex. 4.1, we have

\[
\ddot{\gamma}(x) = \sin \left( \frac{n\pi}{2l} x \right) \quad \text{for} \quad 0 < x < 2l
\]

\[
\dddot{\gamma}(x) = \ddot{\gamma}(x + l) \quad \text{for} \quad -l \leq x \leq l
\]

\[
= \sin \left[ \frac{n\pi}{2l} (x + l) \right]
\]

\[
= \sin \left( \frac{n\pi}{2l} x + \frac{(n-1)\pi}{2} + \frac{\pi}{2} \right)
\]

\[
= \cos \left( \frac{(2n-1)\pi}{2l} x \right) \quad \text{for } n = 1, 2, \ldots
\]

\[
\dddot{\gamma}(x) = \dddot{\gamma}(x) \bigg|_{0 < x < l}
\]

\[
= \cos \left( \frac{(2n-1)\pi}{2l} x \right)
\]
(b) \[ \frac{T_n''}{c^2 T_n} = -\lambda_n \]

where \( \lambda_n = \left[ \frac{(2n-1)\pi}{2L} \right]^2 \) for \( n = 1, 2, \ldots \)

\[ T_n = A_n \cos \left( \frac{(2n-1)\pi c t}{2L} \right) + B_n \sin \left( \frac{(2n-1)\pi c t}{2L} \right) \]

The series expansion \( u(x, t) \) is

\[ u(x, t) = \sum_{n=1}^{\infty} \left[ A_n \cos \left( \frac{(2n-1)\pi c t}{2L} \right) + B_n \sin \left( \frac{(2n-1)\pi c t}{2L} \right) \right] \cos \left( \frac{(2n-1)\pi x}{2L} \right) \]
Q4.

Consider diffusion inside an enclosed circular tube. Let its length (circumference) be $2l$. Let $x$ denote the arc length parameter where $-l \leq x \leq l$. Then the concentration of the diffusing substance satisfies

$$u_t = k u_{xx} \quad \text{for} \quad -l \leq x \leq l.$$

$$u(-l, t) = u(l, t) \quad \text{and} \quad u_x(-l, t) = u_x(l, t)$$

These are called periodic boundary conditions.

(a) Show that the eigenvalues are $\lambda = \left( \frac{n \pi}{2l} \right)^2$

for $n = 0, 1, 2, 3, \ldots$

(b) Show that the concentration is

$$u(x, t) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n \pi x}{l} + B_n \sin \frac{n \pi x}{l} \right) e^{-\frac{n^2 \pi^2 k t}{l^2}}$$

If: let $u(x, t) = T(t) \psi(x)$

$$\frac{T'}{kT} = \frac{\psi''}{\psi} = -\lambda$$
Let $\lambda$ be a complex number.

Case (i) $-\lambda = \beta^2$ where $\beta \neq 0$

$$\therefore \quad \chi(x) = Ce^{\beta x} + De^{-\beta x}$$

$$\therefore u(-l, t) = u(l, t)$$

$$\Rightarrow \quad Ce^{\beta l} + De^{-\beta l} = Ce^{\beta l} + De^{-\beta l} \quad (i)$$

Also,

$$\therefore \chi'(-l) = \chi'(l)$$

$$\Rightarrow \quad C \beta e^{\beta l} - D \beta e^{-\beta l} = C \beta e^{\beta l} - D \beta e^{-\beta l}$$

$$\Rightarrow \quad Ce^{-\beta l} - De^{\beta l} = Ce^{\beta l} - De^{-\beta l} \quad (ii)$$

(i) + (ii) \quad \Rightarrow \quad e^{-\beta l} = e^{\beta l}

\[ \Rightarrow \quad e^{2\beta l} = 1 \]

Similarly, (i) - (ii) \quad \Rightarrow \quad e^{2\beta l} = 1

\[ \therefore \quad \text{Re}(\beta) = 0 \quad \text{and} \quad 2\beta \text{Im}(\beta) = 2n\pi \quad , \quad n = 1, 2, \ldots \]

\[ \text{Im}(\beta) = \frac{n\pi}{l} \quad , \quad n = 1, 2, \ldots \]

$\delta^9$
\[ \lambda = \left( \frac{n\pi}{d} \right)^2 \]

**case (ii)** \(- \lambda = \beta^2 = 0 \)

\[ \therefore \quad 8(x) = Cx + D \]

\[ \therefore \quad 8(-l) = 8(l) \text{ and } 8'(-l) = 8'(l) \]

\[ \Rightarrow \quad 8(x) = D \]

\[ \therefore \quad \lambda = 0 \text{ is also an eigenvalue.} \]

\[ \lambda = \left( \frac{n\pi}{d} \right)^2 \quad \text{for} \quad n=0, 1, 2, 3, \ldots \]

\[ \begin{align*}
(b) \quad \frac{T_n'}{kT_n} &= -\lambda_n \quad \text{where} \quad \lambda_n = \left( \frac{n\pi}{d} \right)^2 \\
\therefore \quad T_n &= e^{-\left( \frac{n\pi}{d} \right)^2 kt} \\
\frac{8_n''}{8_n} &= -\lambda_n
\end{align*} \]
\[ \begin{align*}
\text{for } n \geq 1, \\
\Xi_n &= C_n \cos \left( \frac{n\pi}{l} x \right) + D_n \sin \left( \frac{n\pi}{l} x \right) \\
\text{for } n = 0 \\
\Xi_0 &= D_0
\end{align*} \]

The series expansion \( u(x, t) \) is

\[ u(x, t) = \frac{1}{2} D_0 + \sum_{n=1}^{\infty} \left( C_n \cos \frac{n\pi x}{l} + D_n \sin \frac{n\pi x}{l} \right) e^{-\left(\frac{n\pi}{l}\right)^2 \frac{t}{k}}. \]
On the interval $0 \leq x \leq 1$ of length one, consider the eigenvalue problem

$$-\ddot{x} = \lambda x$$

$$\dot{x}(0) + x(0) = 0 \quad \text{and} \quad x(1) = 0$$

(absorption at one end and zero at the other).

(a) Find an eigenfunction with eigenvalue zero. Call it $x_0(x)$.

(b) Find an equation for the positive eigenvalues $\lambda = \beta^2$.

If $\lambda = 0$

$$\ddot{x} = 0 \quad \Rightarrow \quad x(x) = Ax + B$$

$$\dot{x}(0) + x(0) = 0 \quad \text{and} \quad x(1) = 0$$

$$\Rightarrow \quad A + B = 0$$

$$\therefore \quad x(x) = Ax - A = A(x - 1)$$

$$\Rightarrow \quad x_0(x) = x - 1$$
(b) \(-\ddot{x} = \beta^2 x\) where \(\beta \neq 0\)

\[\Rightarrow x = A \cos \beta x + B \sin \beta x.\]

\[\dot{x}(0) + x(0) = 0 \quad \text{and} \quad x(1) = 0\]

\[\Rightarrow B \beta + A = 0 \quad \text{and} \quad A \cos \beta + B \sin \beta = 0\]

\[\Rightarrow A = -B \beta \neq 0 \quad \text{(why?)}\]

and \(-B \beta \cos \beta + B \sin \beta = 0\)

\[\Rightarrow -\beta \cos \beta + \sin \beta = 0 \quad \text{why}\]

\[\Rightarrow \beta = \tan \beta \quad \text{(why \(\cos \beta \neq 0\?)}\]

: The equation for a true eigenvalue is

\[\tan \beta = \beta\]
Q11
(a) Prove that the (total) energy is conserved for the wave equation with Dirichlet BCs, where the energy is defined to be

\[ E = \frac{1}{2} \int_0^L \left( C^{-2} u_t^2 + u_x^2 \right) \, dx \]

(Compare this definition with Section 2.2)

(b) Do the same for Neumann BCs.

(c) For the Robin BCs, show that

\[ E_R = \frac{1}{2} \int_0^L \left( C^{-2} u_t^2 + u_x^2 \right) \, dx + \frac{1}{2} a_2 \left[ u(l, t) \right]^2 + \frac{1}{2} a_0 \left[ u(0, t) \right]^2 \]

is conserved. Thus, while the total energy \( E_R \) is still a constant, some of the internal energy is lost to the boundary if \( a_0 \) and \( a_2 \) are positive and "gained" from the boundary if \( a_0 \) and \( a_2 \) are negative.
(a) \[
\frac{d}{dt} E(t) = \frac{1}{2} \int_0^l \frac{d}{dt} (c^2 u_t^2 + u_x^2) \, dx
\]
\[
= \frac{1}{2} \int_0^l 2 u_t u_{tt} c^2 + 2 u_x u_{xt} \, dx
\]
\[
= \int_0^l u_t u_{xx} + u_x u_{xt} \, dx \quad (\because u_{tt} = c^2 u_{xx})
\]
\[
= \int_0^l \frac{d}{dx} (u_t u_x) \, dx
\]
\[
= u_t(l,t) u_x(l,t) - u_t(0,t) u_x(0,t)
\]
\[
= 0 \quad \because u(l,t) = u(0,t) = 0 \text{ for } \forall t
\]
\[
\Rightarrow u_t(l,t) = u_t(0,t) = 0
\]
\[\therefore E(t) \text{ is a constant along time.}\]

(b) \[
\frac{d}{dt} E(t) = u_t(l,t) u_x(l,t) - u_t(0,t) u_x(0,l)
\]
in the Neuman B.C's, we have \[u_x(l,t) = u_x(0,t) = 0\]
\[\therefore E'(t) = 0\]
\[ E(t) \] is a constant along time \( t \).

\[ \frac{dE_R}{dt} = \frac{1}{2} \int_0^L \frac{d}{dt} \left( c^{-2} u_t^2 + u_x^2 \right) dx + \]
\[ \frac{1}{2} \frac{d}{dt} a \left[ u(l,t) \right]^2 + \frac{1}{2} \frac{d}{dt} a_0 \left[ u(0,t) \right]^2 \]

\[ = u_t(l,t)u_x(l,t) - u_t(0,t)u_x(0,t) + \]
\[ a_\xi \ u(l,t) \ u_t(l,t) + a_0 \ u(0,t) u_t(0,t) \]

\[ = u_t(l,t) \left( u_x(l,t) + a_\xi \ u(l,t) \right) + \]
\[ u_t(0,t) \left( -u_x(0,t) + a_0 \ u(0,t) \right) \]

\[ = 0 \]

\[ E_R(t) \] is a constant along time \( t \).