Name: _______________________________ PID: ____________________

TA: ____________________ Sec. No: _____ Sec. Time: _____

Math 20A.
Final Examination
December 10, 2009

Turn off and put away your cell phone.
No calculators or any other electronic devices are allowed during this exam.
You may use one page of notes, but no books or other assistance during this exam.
Read each question carefully, and answer each question completely.
Show all of your work; no credit will be given for unsupported answers.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If any question is not clear, ask for clarification.

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1. (6 points) Use the intermediate value theorem to show that the equation $\cos(2x) = x$ has a solution in the interval $(0, \pi/4)$.

Let $f(x) = \cos(2x) - x$

(i) $f(x)$ is continuous on $[0, \pi/4]$

(ii) $f(0) = 1 - 0 = 1 > 0$

(iii) $f(\pi/4) = \cos(\pi/2) - 1$

$= 0 - 1$

$= -1$

$< 0$

By the intermediate value theorem, there exists $x_0 \in (0, \pi/4)$ such that $f(x_0) = 0$

i.e. $\cos(2x_0) = x_0$
2. (6 points) Find the slope of the line tangent to the curve \( x^y = y^x \) at the point \((2,4)\).

\[ x^y = y^x \]

\[ y \ln x = x \ln y \]

\[ \therefore \quad \frac{d}{dx} (y \ln x) = \frac{d}{dx} (x \ln y) \]

\[ \therefore \quad \frac{dy}{dx} \ln x + \frac{x}{y} = \ln y + \frac{1}{y} \frac{dy}{dx} \]

Sub \( x=2 \) and \( y=4 \), we have

\[ \frac{dy}{dx} \ln 2 + \frac{4}{2} = \ln 4 + \frac{1}{4} \frac{dy}{dx} \]

\[ \frac{dy}{dx} (\ln 2 - \frac{1}{4}) = 2 \ln 2 - 2 \]

\[ \frac{dy}{dx} = \frac{2 \ln 2 - 2}{\ln 2 - \frac{1}{4}} \]

\[ = \frac{8 (\ln 2 - 1)}{4 \ln 2 - 1} \]

The slope of the line tangent at \((2,4)\)

\[ \text{is} \quad \frac{8 (\ln 2 - 1)}{4 \ln 2 - 1} \]
3. (6 points) Let \( f(x) \) be a function defined over \([0, 4]\) whose graph is shown below. The graph of \( f(x) \) over \([1, 3]\) is a semicircle centered at \((2, 0)\) with radius 1, and the other parts of the graph are straight lines.

![Graph of f(x) and J(x)](image)

(a) (6 points) Determine \( \int_0^4 f(x) \, dx \) geometrically.

\[
\int_0^4 f(x) \, dx = \text{area of } C + \text{area of } B - \text{area of } A
\]

\[
= \frac{1}{2} (2)(1) + \frac{\pi (1)^2}{2} - \frac{1}{2} (2)(1)
\]

\[
= \frac{\pi}{2}
\]

(b) Let \( F(x) = \int_0^x f(t) \, dt \).

i. Find \( F'(2) \).

\[ F'(x) = f(x) \]

\[ \therefore F'(2) = f(2) = 1 \]

ii. Find \( F'(3) \).

\[ F'(3) = f(3) = 0 \]
4. (6 points) A spherical weather balloon is being inflated at the rate of 12 cubic feet per second. What is the radius of the balloon when its surface area is increasing at a rate of 8 square feet per second?

Note: The formulas for the volume \( V \) and surface area \( A \) of a sphere of radius \( r \) are \( V = \frac{4}{3} \pi r^3 \) and \( A = 4 \pi r^2 \).

**Step 1** find \( \frac{dr}{dt} \) from the formula of \( V 
\[ V = \frac{4}{3} \pi r^3 \]
\[ \frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt} \]
\[ 12 = 4 \pi r^2 \frac{dr}{dt} \]
\[ \frac{dr}{dt} = \frac{3}{\pi r^2} \] \hfill (1)

**Step 2** find \( \frac{dr}{dt} \) from the formula of \( A \)
\[ A = 4 \pi r^2 \]
\[ \frac{dA}{dt} = 8 \pi r \frac{dr}{dt} \]
\[ 8 = 8 \pi r \frac{dr}{dt} \]
\[ \frac{dr}{dt} = \frac{1}{\pi r} \] \hfill (2)

**Step 3** By (1) & (2), we have
\[ \frac{3}{\pi r^2} = \frac{1}{\pi r} \]
\[ \Rightarrow \quad r = 3 \text{ feet.} \]
5. (6 points) Find the values of $a$ and $b$ for which the function

$$f(x) = \begin{cases} \ ax^2 + bx + 2 & \text{if } x \leq 1, \\ -ax^4 - bx^2 & \text{if } x > 1. \end{cases}$$

is differentiable for all real numbers $x$.

\begin{align*}
\text{for } & x < 1, \quad f'(x) = 2ax + b \\
\text{for } & x > 1, \quad f'(x) = -4ax^3 - 2bx \\
\text{for } & x = 1, \quad f(1) = a + b + 2 \\
\text{for } & x \to 1^+, \quad \lim_{x \to 1^+} f(x) = -a - b
\end{align*}

\text{continuity of } f' \quad \Rightarrow \quad a + b + 2 = -a - b \\
\Rightarrow \quad 2a + 2b + 2 = 0 \\
\Rightarrow \quad a + b + 1 = 0 \quad (1)

\text{continuity of } f' \quad \Rightarrow \quad 2a + b = -4a - 2b \\
\Rightarrow \quad 6a + 3b = 0 \\
\Rightarrow \quad 2a + b = 0

\therefore \quad & \text{Sub. } b = -2a \quad \text{to } (1) \\
& \text{we have } \quad a = 1 \\
& \therefore \quad b = -2.

\therefore \quad a = 1, \ b = -2 \quad \text{make } f(x) \text{ is differentiable.
6. (8 points) Let \( f(x) = x^3 - 27x + 5 \)

(a) Find the interval(s) where \( f \) is increasing and the intervals where \( f \) is decreasing.

\[
f'(x) = 3x^2 - 27
\]

\[
f'(x) = 0 \iff 3x^2 - 27 = 0 \iff x = 3 \text{ or } -3
\]

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<th>( x &gt; 3 )</th>
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(b) Find the local maximum and local minimum value(s) of \( f \).

At \( x = -3 \), \( f(-3) = -27 + 81 + 5 = 59 \)

\( \rightarrow \text{a local max} \)

At \( x = 3 \), \( f(3) = 27 - 81 + 5 = -49 \)

\( \rightarrow \text{a local min} \)

(c) Find the intervals where the graph of \( f \) is concave up and the intervals where the graph of \( f \) is concave down.

\[
f''(x) = 6x
\]

\[
f''(x) = 0 \iff x = 0
\]

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<th>( x &lt; 0 )</th>
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(d) Determine the inflection points of the graph of \( f \).

\((0, f(0)) = (0, 5) \) is an inflection point of \( f \).
7. (6 points) Evaluate the following limits.

(a) \[ \lim_{x \to 0} \frac{\tan(\pi x)}{\ln(1 + x)} \]

\[
\lim_{x \to 0} \frac{\tan(\pi x)}{\ln(1 + x)} = \lim_{x \to 0} \frac{\pi \sec^2(\pi x)}{-1/(1 + x)} \\
= \lim_{x \to 0} (1 + x) \pi \sec(\pi x) \quad \text{(Note: } \sec x = \frac{1}{\cos x}) \\
= \pi
\]

(b) \[ \lim_{x \to 0} x^2 \ln|x| \]

\[
\lim_{x \to 0} x^2 \ln|x| = \lim_{x \to 0} \frac{\ln|x|}{x^{-2}} \\
= \lim_{x \to 0} \frac{1}{|x|} \cdot \frac{2}{-x^3} \\
= \lim_{x \to 0} -\frac{x^2}{2} \left( \frac{x}{|x|} \right) \\
= 0 \quad \text{(} \because \left| \frac{x}{|x|} \right| \leq 1 \text{)}
\]
8. (8 points) Compute the derivatives of the following functions.

(a) \( f(x) = (x^4 - 3x^2 + 6)^3 \)

\[
f'(x) = 3(x^4 - 3x^2 + 6)^2 (4x^3 - 6x)
\]

(b) \( f(x) = \frac{x}{3-x^2} \)

\[
f'(x) = \frac{(3-x^2) - x(-2x)}{(3-x^2)^2} = \frac{3+x^2}{(3-x^2)^2}
\]

(c) \( f(x) = 6x(x^2 - 6)^{\frac{1}{3}} + \pi^2 \)

\[
f'(x) = 6 \left( x^2 - 6 \right)^{\frac{1}{3}} + 6x \left( \frac{1}{3} \right) \left( x^2 - 6 \right)^{\frac{-2}{3}} (2x)
\]

\[
= 6 \left( x^2 - 6 \right)^{\frac{1}{3}} + \frac{4}{3} x^2 \left( x^2 - 6 \right)^{\frac{-2}{3}}
\]

(d) \( f(x) = \int_0^x \sqrt{3+t^3} \, dt \)

\[
f'(x) = \sqrt{3+x^3}
\]
9. (8 points) A triangle in the first quadrant is formed by the $x$ and $y$ axes and a line passing through the point $(3, 8)$.

(a) What is the minimum possible area for such a triangle?

Let $a$ and $b$ be a base and height of the triangle. By a similar triangle property, 

$$\frac{3}{a} = \frac{b-8}{b}$$

$$a = \frac{3b}{b-8}$$

The area of the large triangle $A$ is 

$$A = \frac{1}{2} a b$$

$$= \frac{1}{2} \cdot \frac{3b^2}{b-8}$$

$$= \frac{3}{2} \left( \frac{b^2 - 16b}{(b-8)^2} \right)$$

$$A'(b) = 0 \iff b = 0 \text{ (rej)} \text{ or } 16$$

(b) What line gives this minimum area? By the first derivative, $A(16)$ is min and $A'(16) = 48$.

The line to give the minimum area is

$$\frac{y-8}{x-3} = \frac{b-0}{0-a} = \frac{-16}{6}$$

$$y-8 = -\left(\frac{16}{6}\right)(x-3)$$

$$y = -\left(\frac{16}{6}\right)(x-3) + 8$$

$$= -\frac{8}{3}x + 16$$