I did practice 2
Q2 (i) & Q8 (iii)

I did practice 3
Q4 (iii), Q6 (iv) & Q8 (vii & viii)
Mid-term 2 Practice Problem.

Q2

Compute \( \int_{0}^{4} \frac{1}{x^{\frac{3}{5}}} \, dx \) and determine if it converges or not.

pf: \[ \int_{0}^{4} \frac{1}{x^{\frac{3}{5}}} \, dx \]

\[ = \int_{0}^{4} x^{-\frac{3}{5}} \, dx \]

\[ = 5 x^{\frac{2}{5}} \Big|_{0}^{4} \]

\[ = 5 (4)^{\frac{2}{5}} \]

\[ \therefore \text{It converges} \]
QF.

Find the $4^{th}$ roots of $z = -1 - \sqrt{3}i$.

pf: Convert $z$ to be a polar form.
\[ r = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2 \]
\[ \tan \theta = -\sqrt{3} \quad \text{and} \quad z \text{ lies in the third quadrant} \]
\[ \therefore \quad \theta = \frac{\pi}{3} + \pi \]
\[ \therefore \quad z = 2e^{i\left(\frac{\pi}{3} + \pi\right)} = 2e^{i\frac{4\pi}{3}} \]

i. By the formula of finding $k^{th}$ roots
i.e. $x^k = r e^{i\theta} \Rightarrow x = \left\{ x^k e^{i\frac{\theta}{k} + \frac{2\pi ni}{k}} \right\}_{n=0,1,2, \ldots, k-1}$

We have $4^{th}$ roots of $z = 2e^{i\frac{4\pi}{3}}$ are
\[ 2^{\frac{1}{4}} e^{i\left(\frac{4\pi}{3} \times \frac{1}{4}\right)} \quad , \quad 2^{\frac{1}{4}} e^{i\left(\frac{4\pi}{3} \times \frac{1}{4} + \pi\right)} \]
\[ 2^{\frac{1}{4}} e^{i\left(\frac{4\pi}{3} \times \frac{1}{4} + \pi\right)} \quad , \quad 2^{\frac{1}{4}} e^{i\left(\frac{4\pi}{3} \times \frac{1}{4} + \frac{3\pi}{2}\right)} \]
\[ = 2^{\frac{1}{4}} \left( -\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \quad , \quad 2^{\frac{1}{4}} \left( \frac{-\sqrt{3}}{2} + i \frac{1}{2} \right) \]
\[ 2^{\frac{1}{4}} \left( -\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) \quad , \quad 2^{\frac{1}{4}} \left( \frac{-\sqrt{3}}{2} - i \frac{1}{2} \right) \]

\[ \therefore \quad k \]
Practice Problem 3

Q.4. Determine if the series \( \sum \frac{\cos(\frac{\pi}{2} + n\pi)}{\sqrt{n}} \) converges absolutely or conditionally.

Pf: Does \( \sum \left| \frac{\cos(\frac{\pi}{2} + n\pi)}{\sqrt{n}} \right| \) converge?

\[
\sum \left| \frac{\cos(\frac{\pi}{2} + n\pi)}{\sqrt{n}} \right| = \sum \frac{|\cos(\frac{\pi}{2})|}{\sqrt{n}} = \sum \frac{1}{\sqrt{n}}
\]

By P-test, \( \sum \left| \frac{\cos(\frac{\pi}{2} + n\pi)}{\sqrt{n}} \right| \) diverges.

\[ \therefore \sum \frac{\cos(\frac{\pi}{2} + n\pi)}{\sqrt{n}} \text{ can't converge absolutely} \]

Does \( \sum \frac{\cos(\frac{\pi}{2} + n\pi)}{\sqrt{n}} \) converge?

\[
\sum \frac{\cos(\frac{\pi}{2} + n\pi)}{\sqrt{n}} = \sum \frac{(-1)^n(\frac{1}{\sqrt{n}})}{\sqrt{n}} = \frac{1}{\sqrt{2}} \sum (-1)^n
\]

As \( \sum \) the sequence \( \sum \frac{(-1)^n}{n} \) is an alternating series.
2. \( \frac{(-1)^n}{\sqrt{n}} \to 0 \) as \( n \to \infty \)

3. \( \left| \frac{(-1)^n}{\sqrt{n}} \right| \) is monotone decreasing.

By Leibniz Test,

\[ \sum \frac{\cos \left( \frac{\pi}{4} + n \pi \right)}{\sqrt{n}} \] converge.

It means \( \sum \frac{\cos \left( \frac{\pi}{4} + n \pi \right)}{\sqrt{n}} \) converge conditionally.
Q6. Find the interval of convergence of \( \sum \frac{2^n}{n!} x^n \).

If the interval is finite, can you determine if it converges at the end-points?

**Pf:** let \( a_n = \frac{2^n}{n!} x^n \)

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{2^{n+1}}{(n+1)!} x^{n+1} \cdot \frac{n!}{2^n} \frac{1}{x^n} \right| = \frac{2}{n+1} |x|
\]

\[
\therefore \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1
\]

\[
\Rightarrow \lim_{n \to \infty} \frac{2}{n+1} |x| < 1
\]

\[
\Rightarrow |x| < 1
\]

\[
\Rightarrow |x| \text{ can be any number.}
\]

\[
\Rightarrow |x| < +\infty
\]

\[
\Rightarrow \text{By the Ratio test, the interval of convergence is the whole Real line, i.e. } |x| < +\infty, \text{ or } (-\infty, \infty)
\]
Q6. Find the interval of convergence of \( \sum \frac{n^6}{n^4+1} (x-3)^n \).

If the interval is finite, can you determine if it converges at the endpoints?

\[ P_f: \quad \text{let } a_n = \frac{n^6}{n^4+1} (x-3)^n \]

\[ \frac{|a_{n+1}|}{a_n} = \left| \frac{(n+1)^6}{(n+1)^4+1} \frac{(x-3)^{n+1}}{n^6 \frac{n^6+1}{(x-3)^n}} \right| \]

\[ = \left| \frac{(n+1)^6}{(n+1)^4+1} \frac{n^6+1}{n^6} \right| (x-3)^n \]

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x-3| \]

\[ \therefore \quad |x-3| < 1 \]

By the Ratio Test, the interval of convergence is \( (2, 4) \).

At \( x=4 \), \( a_n = \frac{n^6}{n^4+1} \) at \( x=4 \).
\[
\text{let } \quad b_n = \frac{1}{n^2}
\]

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^6}{n^3 + 1} \times n^2 = \lim_{n \to \infty} \frac{n^8}{n^6 + 1} = 1
\]

And \( a_n > 0 \) & \( b_n > 0 \)

(i) By the Limit Comparison Test, the convergence of \( \sum b_n \) \( \Rightarrow \) the convergence of \( \sum a_n \), and

(ii) By the \( p \)-test, \( \sum b_n \) converge

(iii) \( \sum a_n \) converges, at \( x = 4 \)

At \( x = 2 \), \( a_n = \frac{(-1)^n n^6}{n^3 + 1} \) at \( x = 2 \)

By (i) \( \sum a_n \) is alternating series at \( x = 2 \)

(ii) \( a_n \to 0 \) as \( n \to \infty \) at \( x = 2 \)

(iii) \( |a_n| \downarrow \) at \( x = 2 \)

(i) By Leibniz Test, \( \sum a_n \) converges at \( x = 2 \).
Q7. Find the Taylor Series of the function \( f(x) = \sin x \), \( c = \pi \).

**Proof:**

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!} (x-\pi)^n \]

where \( f(x) = \sin x \)

\[ f^{(n)}(x) = \frac{d^n}{dx^n} \sin x = \begin{cases} \frac{(-1)^{n+1}}{n} \cos x & \text{if } n \text{ is odd} \\ \frac{(-1)^n}{n} \sin x & \text{if } n \text{ is even} \end{cases} \]

\[ f^{(n)}(\pi) = \begin{cases} (-1)^{n+1}(-1) = (-1)^{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \]

\[ \therefore \frac{\sin x}{\cos x} = \sum_{n=0}^{\infty} \frac{(-1)^{\frac{n+1}{2}}}{n!} (x-\pi)^n \]

\[ = \sum_{m=0}^{\infty} \frac{(-1)^{\frac{2m+1}{2}}}{(2m+1)!} (x-\pi)^{2m+1} \]

where \( n = 2m+1 \) is odd.

\[ = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(2m+1)!} (x-\pi)^{2m+1} \]
Q.8 Find the general solution to the differential equation.

(i) \( 2y' + 5y = 4 \) , (ii) \( yy' = x \)

PF (i) Use a formula (4) on p. 529 in yr textbook.

\[ y' + \frac{5}{2}y = 2 \]

\[ A(x) = \frac{5}{2} , \quad B(x) = 2 \]

\[ \therefore y = \frac{1}{e^{\frac{5}{2}x}} \left( 0 \ e^{\frac{5}{2}x} 2 \ dx + C \right) \]

\[ = \frac{4}{5} + Ce^{\frac{5}{2}x} \]

(ii) \( yy' = x \) Use separation of variables

\[ y \frac{dy}{dx} = x \]

\[ y \ dy = x \ dx \]

\[ \int y \ dy = \int x \ dx \]

\[ \frac{y^2}{2} = \frac{x^2}{2} + C \]

\[ y^2 = x^2 + C \]

p.9