Ex 14.7

Q14

Find the critical points of \( f(x, y) = e^{x^2 - y^2 + y} \) and determine its behavior by second derivatives test.

**Proof:**

1. \( \frac{\partial f}{\partial x} = 2x e^{x^2 - y^2 + y} \quad \text{--- (i)} \)

2. \( \frac{\partial^2 f}{\partial y^2} = (-2y + 4) e^{x^2 - y^2 + y} \quad \text{--- (ii)} \)

(i) \( = 0 \Rightarrow x = 0 \)

If \( x = 0 \) and sub it to (ii):

\( \Rightarrow -2y + 4 = 0 \Rightarrow y = 2 \).

\( (x, y) = (0, 2) \) is a critical point.

Since \( \frac{\partial^2 f}{\partial x^2} = 2 e^{x^2 - y^2 + y} (1 + 2x^2) \)

\( \frac{\partial^2 f}{\partial x \partial y} = 4x (2-y) e^{x^2 - y^2 + y} \)
\[
\frac{\partial^2 f}{\partial y^2} = 2e^{x^2-y^2+4y}(2y^2+8y+7)
\]

\[D(0,2) = f_{xx}f_{yy} - f_{xy}^2 |_{(x,y)=(0,2)} = -4e^8 < 0\]

By the second derivative test, it is a saddle point.

Q24
Show that \( f(x,y) = x^2 \) has infinitely many critical points, and that the second derivative test fails for all of them. What is the minimum value of \( f \)? Does \( f(x,y) \) have any local maxima?

\[\begin{align*}
f &= 0 = \frac{\partial f}{\partial x} = 2x \\
0 &= \frac{\partial f}{\partial y} = 0
\end{align*}\]

\[\therefore (0, y) \text{ are critical points for any } y.\]
\[
\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 0
\]

i. \( D(0, y) = f_{xx} - f_{xy} \bigg|_{(x,y)=(0,y)} = 0 \)

ii. The Second Derivative Test fails.

However, we know \( f(x, y) = x^2 \) is always non-negative and reach 0 at \((0, y)\).

iii. The minimum value of \( f \) is 0.

And there are no local maxima since for any \((x, y)\) in the coordinate plane.

- if \( x > 0 \), then \( f(x+\Delta x, y) = (x+\Delta x)^2 \geq x^2 \)
  \( \Delta x > 0 \)

- if \( x < 0 \), then \( f(x-\Delta x, y) = (x-\Delta x)^2 \geq x^2 \)
  \( \Delta x > 0 \)

iv. There are no local maxima.
Q38.
Determine the global extreme values of the function
\[ f(x, y) = 5x - 3y, \quad y \geq x - 2, \quad y \geq -x - 2, \quad y \leq 3. \]

If:

\[ D = \{ (x, y) \mid y \geq x - 2, \quad y \geq -x - 2, \quad y \leq 3 \} \]

\[ \frac{\partial f}{\partial x} = 5 \]
\[ \frac{\partial f}{\partial y} = -3 \]

There are no critical point in \( D \).

We study \( f(x, y) \) along boundary of \( D \).

(i) \( f \) along \( y = 3 \) and \(-5 \leq x \leq 5\)
\[ f(x, 3) = 5x - 9 \]

It is clear that
\[ f(-5, 3) = -34 \quad \text{is a local min} \]
\[ f(5, 3) = 16 \quad \text{is a local max} \]
(ii) For any \( y = x - 2 \), \( 0 \leq x \leq 5 \)

\[
f(x, x-2) = 5x - 3(x-2) = 2x + 2
\]

It is clear that
\( f(0, -2) = 6 \) is a local min
\( f(5, 3) = 16 \) is a local max

(iii) For any \( y = -x - 2 \), \( -5 \leq x \leq 0 \)

\[
f(x, -x-2) = 5x - 3(-x-2) = 8x + 2
\]

It is clear that
\( f(0, -2) = 6 \) is a local max
\( f(-5, 3) = -34 \) is a local min

By comparing all local max & min,
we have
\( f(-5, 3) = -34 \) is a global min on \( D \)
\( f(5, 3) = 16 \) is a global max on \( D \).
Q40. Determine the global extreme values of the function.

\[ f(x, y) = x^2 + x^2 y + 2y^2, \quad x, y \geq 0, \quad x+y \leq 1. \]

\[ D = \{(x, y) \mid x \geq 0, \quad y \geq 0, \quad x+y \leq 1\} \]

\[ \begin{align*}
\frac{\partial f}{\partial x} &= 3x^2 + 2xy & \text{(i)} \\
\frac{\partial f}{\partial y} &= x^2 + 4y & \text{(ii)} \\
\text{from (ii),} & \quad y = \frac{x^2}{4} \\
\text{Sub } y = \frac{x^2}{4} \quad \text{to (i),} & \quad 3x^2 + 2x\left(\frac{x^2}{4}\right) = 0 \\
& \quad x^2 \left(3 + \frac{x}{2}\right) = 0 \\
& \quad x = 0 \quad \text{or} \quad x = -6.
\end{align*} \]
\[ \frac{1}{3} \text{ if } x=0, \quad y=0 \]
\[ \frac{1}{3} \text{ if } x=-6, \quad y=9. \]

However, \((0,0)\) and \((-6,9)\) are not in the interior of \(D\).

There are no critical points in \(D\). Then, we study its boundary behavior.

(i) Along \(x=0\) \(0 \leq y \leq 1\)
\[ f(0, y) = 2y^2. \]

It is clear that
\[ f(0,0) = 0 \text{ is a local min} \]
\[ f(0,1) = 2 \text{ is a local max} \]

(ii) Along \(y=0\) \(0 \leq x \leq 1\)
\[ f(x,0) = x^3 \]

It is clear that
\[ f(0,0) = 0 \text{ is a local min} \]
\[ f(1,0) = 1 \text{ is a local max} \]

\[ \text{P.T.} \]
(iii) Along $x+y=1$, $0 \leq x \leq 1$

$$f(x, -x) = x^3 + x^2(-x) + 2(-x)^2$$
$$= x^3 + x^2 - x^3 + 2 - 4x + 2x^2$$
$$= 3x^2 - 4x + 2$$

Let $g(x) = f(x, 1-x) = 3x^2 - 4x + 2$

$$0 = g'(x) = 6x - 4$$

$$x = \frac{2}{3}$$ is a critical point

$$g\left(\frac{2}{3}\right) = f\left(\frac{2}{3}, \frac{1}{3}\right) = 3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 2 = \frac{2}{3}$$

$$f(0, 1) = 2$$

$$f(1, 0) = 1$$

$$f(0, 1) = 2$$ is a local max.

$$f\left(\frac{2}{3}, \frac{1}{3}\right) = \frac{2}{3}$$ is a local min.

By comparing all local min & max, we have

$$f(0, 1) = 2$$ is a global max in $D$

$$f(0, 0) = 0$$ is a global min in $D$
Ex 14.8

Q15

Find the point \((a, b)\) on the graph of \(y = e^x\) where the value \(ab\) is as small as possible.

**Proof:** Let \(f(x, y) = xy\) be the function you want to minimize.

Let \(g(x, y) = y - e^x\) be the constraint.

\[
\nabla f = \lambda \nabla g
\]

\[
y = -xe^x \quad \text{(i)}
\]

\[
x = \lambda \quad \text{(ii)}
\]

\[
0 = y - e^x \quad \text{(iii)}
\]

from (ii) \(x = \lambda\)

from (iii) \(y = e^x\)

plug \(x = \lambda\) & \(y = e^x\) to (i)

\[
e^x + xe^x = 0
\]

\[
\Rightarrow e^x(1 + x) = 0
\]

\[
\Rightarrow x = -1
\]

\[
\therefore y = e^{-1}
\]

\[
\therefore (-1, e^{-1}) \text{ is a critical point.}
\]
and as $x \to \infty$, $y = e^x \to \infty$.

$\therefore xy \to \infty$

as $x \to -\infty$, $y = e^x \to 0$

and by L'Hopital's rule,

we have $xe^x \to 0$ as $x \to 0$

\begin{equation}
\text{maximum is subject to } y = e^x \text{ has min value } -e^{-1} \text{ at } (-1, e^{-1}) .
\end{equation}

Q24.

Show that the maximum value of $f(x,y) = x^2y^3$ on the unit circle is $\frac{6}{25} \sqrt{5}$.

\textbf{Pf: } Let $g(x,y) = x^2 + y^2 - 1$ be a constraint.

\begin{align*}
\nabla f &= \lambda \nabla g \\
2xy^3 &= 2\lambda x \\
3x^2y^2 &= 2\lambda y \\
x^2 + y^2 &= 1
\end{align*}
From (i)
\[ 2xy^3 - 2\lambda x = 0 \]
\[ x(y^3 - \lambda) = 0 \]
\[ \Rightarrow x = 0 \quad \text{or} \quad y^3 = \lambda. \]

**Case 1.** \( x = 0 \)
Sub \( x = 0 \) to (iii)
\[ \Rightarrow y = 1 \quad \text{or} \quad -1 \]
and sub \( y = \pm 1 \) \( \text{and} \) \( x = 0 \) to (ii)
\[ \Rightarrow \lambda = 0 \]
i.e. \((0, 1)\) and \((0, -1)\) are critical points.

**Case 2.** \( \lambda = y^3 \)
Sub \( \lambda = y^3 \) to (ii)
\[ \Rightarrow 3xy^2 = 2y^3 \cdot y \]
\[ y^2(3x - 2y^2) = 0 \]
\[ \Rightarrow y = 0 \quad \text{or} \quad x = \sqrt[3]{\frac{2}{3}}y \quad \text{or} \quad x = -\sqrt[3]{\frac{2}{3}}y \]
case 2.1 \[ y = 0 \]

Sub \( y = 0 \) to (iii)

\[ \Rightarrow x = \pm 1 \]

\[ \therefore (1, 0) \text{ and } (-1, 0) \text{ are critical points} \]

case 2.2 \[ x = \sqrt{\frac{2}{3}} y \]

Sub \( x = \sqrt{\frac{2}{3}} y \) to (iii)

\[ \therefore \begin{cases} \frac{2}{3} y^2 + y^2 = 1 \\ \frac{5}{3} y^2 = 1 \end{cases} \]

\[ \Rightarrow \begin{cases} y = \sqrt{\frac{3}{5}} \text{ or } -\sqrt{\frac{3}{5}} \end{cases} \]

\[ \therefore \left( \sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}} \right) \text{ and } \left( -\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}} \right) \text{ are critical points} \]
case 2.3 \( x = -\sqrt{\frac{2}{3}} y \).

Substitute \( x = -\sqrt{\frac{2}{3}} y \) to (ciii):

\[
\therefore \quad \frac{2}{3} y^2 + y^2 = 1
\]

\[
\Rightarrow \begin{cases} 
  y = \sqrt{\frac{3}{5}} \\
  y = -\sqrt{\frac{3}{5}}
\end{cases}
\]

\[
\Rightarrow (-\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}) \quad \text{and} \quad (\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}})
\]

are critical points.

By comparing the values at all critical points:

\((1, 0), \ (0, 1), \ (0, -1), \ (-1, 0), \ (0, 1), \ (-\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}), \ (\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}), \ (\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}), \ (-\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}})\)

\( f(\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}) \) and \( f(-\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}) \) both gives

\[
\frac{6}{25} \sqrt{\frac{3}{5}}
\]

which is the max. value. \( \star \)