Ex. 2.1

11. Sketch the level surface and a section of the graph of \( f(x, y, z) = -x^2 - y^2 - z^2 \)

(i) The level surface first, i.e. \( f(x, y, z) = c \) where \( c \) is a constant

\[ x^2 + y^2 + z^2 = -c \]

\[ \{ (x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = c \} \]

is a level set and the set is \{ \}

- a sphere if \( c < 0 \)
- the origin if \( c = 0 \)
- empty if \( c > 0 \)

\[ \text{if } c < 0 \quad \text{if } c = 0 \quad \text{if } c > 0 \]
(ii) sections of the graph.

There are three sections corresponding to the graph \( \cap \mathbf{P}_{xy} \), the graph \( \cap \mathbf{P}_{xz} \)
the graph \( \cap \mathbf{P}_{zx} \)

where \( \mathbf{P}_{xy} \) is a xy plane \( (z \text{ is fixed}) \)
\( \mathbf{P}_{xz} \) is a xz plane \( (y \text{ is fixed}) \)
\( \mathbf{P}_{zy} \) is a yz plane \( (x \text{ is fixed}) \)

Let us see the graph \( \cap \mathbf{P}_{xy} \).

i.e. \( z \) is fixed to be \( z_0 \).

\[
\begin{align*}
\mathbf{P}(x, y, z_0) &= -x^2 + y^2 + z_0^2 \\
&= -x^2 - y^2 - z_0^2
\end{align*}
\]

The section is a paraboloid of revolution.
opening down in \((x, y, f(x, y, z_0))\) space.
The graph looks like this:

\[ f(x, y, z) \]

\[ (x, y, z_0) \]

\[ y \]

\[ x \]

\[ -z_0 \]

As \( f(x, y, z) \) is symmetric with respect to \( x, y, z \),

\[ \text{The graph } \nabla P_{xy} = \text{The graph } \nabla P_{zx} = \text{The graph } \nabla P_{zy} \]

is a paraboloid of revolution along its corresponding axis.
Q28. \[ y^2 = x^2 + z^2 \]

Fix \( z = z_0 \), \( y^2 = x^2 + z_0^2 \) is a hyperbola.

Fix \( y = y_0 \), \( y_0^2 = x^2 + z^2 \) is a circle.

Fix \( x = x_0 \), \( y^2 = x_0^2 + z^2 \) is a hyperbola.
The surface of $f$.

For Q21, I think it can be done by a similar method as in Q20.

Q21. I think it can be done by yourself.
Ex. 2.2.

(9c) The limit exists.

By the theorem 3 (iii) & (iv) on P. 116.

\[
\lim_{{(x,y) \to (0,0)}} \frac{xy}{x^2 + y^2 + 2} = \lim_{{(x,y) \to (0,0)}} (xy) \lim_{{(x,y) \to (0,0)}} \frac{1}{x^2 + y^2 + 2} = 0 \times \frac{1}{2} = 0
\]

Remark: You can use the theorem because
(i) \(xy\) & \(x^2 + y^2 + 2\) are continuous
(ii) \(x^2 + y^2 + 2 \to 2 > 0\) as \((x,y)\to (0,0)\)

(10 a) The limit exists.

By the same reason in (9c), we have

\[
\lim_{{(x,y) \to (0,0)}} \frac{e^{xy}}{x + 1} = \lim_{{(x,y) \to (0,0)}} e^{xy} \times \lim_{{(x,y) \to (0,0)}} \frac{1}{x + 1} = 1 \times 1 = 1.
\]
(10b) The limit does not exist.

Consider the path \((x, 0)\) approaching \((0, 0)\)

Informal writing/intuitively thinking, we have

\[
\lim_{x \to 0} \frac{\cos x - 1 - \frac{x^2}{2}}{x^4 + y^4} = \lim_{x \to 0} \frac{\cos x - 1 - \frac{x^2}{2}}{x^4}
\]

\[
= \lim_{x \to 0} \frac{-\sin x - x}{4x^2} \quad \text{(L. Hospital's Rule)}
\]

\[
= \lim_{x \to 0} \frac{-\cos x - 1}{12x^2} \quad \text{(L. Hospital's Rule)}
\]

\[
= \text{undefined } \rightarrow \infty
\]

Formally, we should argue like this

\[
\lim_{x \to 0} \frac{x^4 + y^4}{\cos x - 1 - \frac{x^2}{2}} = 0 \quad \text{(By using L. Hospital's Rule twice)}
\]

(Please do it by yourself)

\[
\therefore \lim_{y \to 0} \frac{\cos x - 1 - \frac{y^2}{2}}{x^4 + y^4} = \lim_{y \to 0} \frac{x^4 + y^4}{\cos x - 1 - \frac{x^2}{2}} \text{ is undefined.}
\]

i. The limit does not exist.
(11a) The question is still wrong.

Because \( \frac{\sin xy}{xy} \) is undefined either \( x \) or \( y \neq 0 \).

So, we should modify the question as

\[
\lim_{(x,y) \to (0,0)} \frac{\sin xy}{xy} = x y.
\]

To find its limit,

we first let \( f(x, y) = xy \)

\[
g(t) = \begin{cases} 
\frac{\sin t}{t} & \text{if } t \neq 0 \\
0 & \text{if } t = 0 
\end{cases}
\]

We know that \( f \) & \( g \) are continuous.

\( \therefore \) By the thm 5 on R121,

we have \( g \circ f \) is continuous.

Indeed, \( g \circ f (x, y) = \frac{\sin xy}{xy} \).

\( \therefore \)

\[
\lim_{(x,y) \to (0,0)} \frac{\sin xy}{xy} = g \circ f (0, 0) = g(0) = 1.
\]
Ex 2.4

(15) Let \( S(t) = (\sin 3t, \cos 3t, 2t^{3/2}) \)

\[
S'(t) = (3 \cos 3t, -3 \sin 3t, 3t^{1/2})
\]

\[
S'(1) = (3 \cos 3, -3 \sin 3, 3)
\]

The equation of the tangent line at \( t = 1 \) is

\[
S'(1) (t-1) + S(1)
\]

\[
= (3(1-1) \cos 3, -3(1-1) \sin 3, 3(1-1)) + (\sin 3, \cos 3, 2)
\]

\[
= (3(1) \cos 3 + \sin 3, -3(1) \sin 3 + \cos 3, 3(1)+2)
\]