Ex. 2.3

Q4a. The domain of $f$ is all points in $\mathbb{R}^2$ except $(0,0)$, i.e. $\mathbb{R}^2 \setminus \{(0,0)\}$ (why?)

\[
\frac{af}{dx} = \frac{2y(x^2+y^2)^2 - 2(x^2+y^2)(2x)(2y)}{(x^2+y^2)^4}
\]

\[
= \frac{2y^5 + 4x^2y^2 - 6x^2y - 8y^3}{(x^2+y^2)^4} \quad \text{on} \quad \mathbb{R}^2 \setminus \{(0,0)\}
\]

\[
\frac{af}{dy} = \frac{2x^5 + 4y^3x^2 - 6y^2x - 8y^2x^3}{(x^2+y^2)^4} \quad \text{on} \quad \mathbb{R}^2 \setminus \{(0,0)\}
\]

Since $\frac{af}{dx}$, $\frac{af}{dy}$ are defined and continuous

on $\mathbb{R}^2 \setminus \{(0,0)\}$. (Do you know why?)

\( \therefore f \text{ is } C^1 \text{ on } \mathbb{R}^2 \setminus \{(0,0)\} \)
4b) \( f(x, y) = \frac{x}{y} + \frac{y}{x} \) has a domain:

\( \{(x, y) \in \mathbb{R}^2 \mid \text{either } x \text{ is zero or } y \text{ is zero}\} \)

On the domain,

\[
\frac{\partial f}{\partial x} = \frac{1}{y} + \left( -\frac{y}{x^2} \right)
\]

\[
\frac{\partial f}{\partial x} = \frac{1}{x} - \frac{x}{y^2}
\]

Both derivatives are well-defined and continuous on the domain. (Do you know why?)

\[ f \in C^1 \]
Q20. Suppose \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a linear map.

What is the derivative of \( f \)?

**Proof:**

\[
\begin{align*}
\frac{\partial f_j}{\partial x_i} &= \lim_{\Delta x_i \to 0} \frac{1}{\Delta x_i} \left( f_j(x_1, \ldots, x_i + \Delta x, \ldots, x_n) - f_j(x_1, \ldots, x_i, \ldots, x_n) \right) \\
&= \lim_{\Delta x_i \to 0} \frac{1}{\Delta x_i} \left( f_j(x_1, \ldots, x_i, \ldots, x_n) + f_j(0, \ldots, \Delta x, \ldots, x_n) \quad \text{\( i^{th} \) position} \\
&\quad - f_j(x_1, \ldots, x_i, \ldots, x_n) \right) \\
&= \lim_{\Delta x_i \to 0} \frac{1}{\Delta x_i} \cdot \Delta x \cdot f_j(0, \ldots, 1, \ldots, 0) \quad \text{\( i^{th} \) position} \\
&= f_j(0, \ldots, 1, \ldots, 0) \quad \text{\( i^{th} \) position}
\end{align*}
\]
The derivative of $f$ is $Df$.

\[
Df = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
f_1(1,0,\ldots,0) & \ldots & f_1(0,\ldots,0,1) \\
\vdots & \ddots & \vdots \\
f_m(1,0,\ldots,0) & \ldots & f_m(0,\ldots,0,1)
\end{bmatrix}
\]
Ex. 2.5.

Q4. On one hand, we find have

\[ h(x, y) = f(u(x, y), v(x, y)) \] 

You substitute \( u(x, y) = e^{-x-y} \), \( v(x, y) = e^{xy} \) to (1)

\[
h(x, y) = f(e^{-x-y}, e^{xy})
\]

\[
= \frac{(e^{-x-y})^2 + (e^{xy})^2}{(e^{-x-y})^2 - (e^{xy})^2}
\] 

Then you compute \( \frac{\partial h}{\partial x} \) and \( \frac{\partial h}{\partial y} \) directly from (2)

On the other hand,

by the chain’s rule,

\[
 \frac{\partial h}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x},
\]

\[
 \frac{\partial h}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}.
\]

You compute \( \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y} \) and \( \frac{\partial v}{\partial y} \)

to find \( \frac{\partial h}{\partial x} \) and \( \frac{\partial h}{\partial y} \)
Now, you have two ways to get $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

What you need to check is if they match each other.

Q8. I just compute $\frac{\partial f}{\partial \rho}$ for you, and you find the rest.

$$\frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \rho} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial \rho}.$$  

$$\frac{\partial x}{\partial \rho} = \cos \theta \sin \phi.$$  

$$\frac{\partial y}{\partial \rho} = \sin \theta \sin \phi.$$  

$$\frac{\partial z}{\partial \rho} = \cos \phi.$$  

$$\therefore \frac{\partial f}{\partial \rho} = \cos \theta \sin \phi \frac{\partial f}{\partial x} + \sin \theta \sin \phi \frac{\partial f}{\partial y} + \cos \phi \frac{\partial f}{\partial z}.$$
\[
\frac{\partial}{\partial s}(f \cdot T)(1, 0)
\]
\[
= \frac{\partial f}{\partial u} \frac{\partial T_1}{\partial s} + \frac{\partial f}{\partial v} \frac{\partial T_2}{\partial s}
\]
where \[
\begin{align*}
T_1(s, t) &= \cos(t^2 s) \\
T_2(s, t) &= \log \sqrt{1 + s^2}
\end{align*}
\]

Since \[
\frac{\partial f}{\partial u} = -\sin u \sin v
\]
\[
\frac{\partial f}{\partial v} = \cos u \cos v
\]
\[
\frac{\partial T_1}{\partial s} = -t^2 \sin(t^2 s)
\]
\[
\frac{\partial T_2}{\partial s} = \frac{1}{\sqrt{1 + s^2}} \times \frac{1}{2} \times \frac{1}{\sqrt{1 + s^2}} \times 2s = \frac{s}{1 + s^2}
\]

\[
A(f(1, 0)) = (1, 0), \quad \Rightarrow (u, v) = (T_1(1, 0), T_2(1, 0)) = (1, \log \sqrt{2})
\]

\[
\frac{\partial}{\partial u}(1, \log \sqrt{2}) = -\sin 1 \sin(\log \sqrt{2})
\]
\[
\frac{\partial}{\partial v}(1, \log \sqrt{2}) = \cos 1 \cos(\log \sqrt{2})
\]
\[
\frac{\partial T_1}{\partial s}(1, 0) = 0, \quad \frac{\partial T_2}{\partial s}(1, 0) = \frac{1}{2}.
\]

\[
\frac{\partial f \circ T}{\partial s}(1, 0) = \frac{1}{2} \cos 1 \cos(\log \sqrt{2})
\]
Q15. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \)

\[
(x, y) \mapsto (e^{x+y}, e^{x-y})
\]

Let \( c(t) \) be a path with \( c(0) = (0, 0) \) and \( c'(0, 0) = (1, 1) \). What is the tangent vector to the image of \( c(t) \) under \( f \) at \( t = 0 \)?

**Proof:** Let \( f_1(x, y) = e^{x+y} \), \( f_2(x, y) = e^{x-y} \).

The question asks what is \((f_1 \circ c)'(0), (f_2 \circ c)'(0)\).

To answer it,

**Note:** \( c(t) : \mathbb{R} \to \mathbb{R}^2 \)

\[
t \mapsto (c_1(t), c_2(t))
\]

where \( c_1(0) = c_2(0) = 0 \), \( c_1'(0) = c_2'(0) = 1 \).

Also, when \( t = 0 \), \((c_1(0), c_2(0)) = (0, 0)\).

\[
\frac{d}{dt} (f \circ c)(0) = \frac{\partial f_1}{\partial x} (0, 0) \frac{dc_1}{dt} (0) + \frac{\partial f_2}{\partial y} (0, 0) \frac{dc_2}{dt} (0)
\]
\[
\begin{align*}
= \frac{\partial f_1}{\partial x}(0,0) + \frac{\partial f_1}{\partial y}(0,0) \\
= 1 + 1 \\
= 2
\end{align*}
\]

Similarly, we have
\[
\left( \frac{d}{dt} f_2 \circ c \right)(0) = \frac{\partial f_2}{\partial x}(0,0) \frac{dc_1}{dt}(0) + \frac{\partial f_2}{\partial y}(0,0) \frac{dc_2}{dt}(0) \\
= 0.
\]

Remark:

There is another expression for chain rule. For example, in our question, it is
\[
\begin{bmatrix}
\nabla f_1(0,0) \cdot c'(0) \\
\nabla f_2(0,0) \cdot c'(0)
\end{bmatrix}
\]
Q16 Let \( f(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \). Compute \( \nabla f(x, y) \)

\[
\frac{\partial f}{\partial x} = -\frac{1}{2} \frac{1}{(x^2 + y^2)^{\frac{3}{2}}} \times 2x
\]

\[
= -\frac{x}{(x^2 + y^2)^{\frac{3}{2}}}
\]  
(1)

Since \( f(x, y) \) is symmetric to \( x \) & \( y \),

i.e. \( f(x, y) = f(y, x) \)

So, we immediately know that

\[
\frac{\partial f}{\partial y} = -\frac{y}{(y^2 + x^2)^{\frac{3}{2}}}
\]

\{(We get (2) by changing \( x \) to \( y \) and \( y \) to \( x \) in (1)).\}

Remark: In fact, we use the same technique to find \( \frac{\partial f}{\partial y} \) in Ex. 2-3 4a & b. Do you get it? \( \& \)
Ex. 2.6

Q: Find a unit normal to the surface

\[ \cos(xy) = e^z - 2 \at \end{align*}

at \((1, \pi, 0)\).

pf. Just like example 7 on p. 170.

Let \( f(x, y, z) = e^z - \cos(xy) - 2 \)

\[ \nabla f (x, y, z) = y \sin(xy) \hat{i} + x \sin(xy) \hat{j} + e^z \hat{k} \]

\[ \therefore \nabla f (1, \pi, 0) = \hat{k} \]

\[ \| \nabla f (1, \pi, 0) \| = 1. \]

\[ \therefore \text{A unit normal is} \quad \frac{\nabla f}{\| \nabla f \|} = \hat{k}. \]

Remark: If your answer is \(-\hat{k}\),

it is still correct.

Do you know why?
Q20. Suppose that a mountain has the shape of an elliptic paraboloid \( z = c - ax^2 - by^2 \), where \( a, b \) and \( c \) are positive constants, \( x \) and \( y \) are the east-west and north-south map coordinates, and \( z \) is the altitude above sea level (\( x, y, z \) are all measured in meters). At the point \((1, 1)\), in what direction is the altitude increasing most rapidly? If a marble were released at \((1, 1)\), in what direction would it begin to roll?

Let \( f(x, y) = z = c - ax^2 - by^2 \) (Altitude function).

Since directional derivative of \( f \) along

\[ \nabla f \cdot \hat{v} = |\nabla f| |\hat{v}| \cos \theta \]

where \( \theta \) is the angle between \( \nabla f \) & \( \hat{v} \).

\[ = |\nabla f| \cos \theta \]
1. We know that the change of the rate

\[ \frac{\partial S}{\partial t} \]

is \( \max. \) (\( \min. \) ) iff \( \frac{\partial^2 S}{\partial t^2} = 0 \) (\( \neq 0 \))

\[ \downarrow \quad \uparrow \]

increase \quad most \quad decrease \quad most \quad rapidly \quad rapidly

\[ \text{largest positive value} \quad \text{largest negative value} \]

2. For the first part,

the direction in which the altitude increases most rapidly is \( \nabla f (1, 1) = (-2a, -2b) \)

For the second part, owing to the gravity, a marble will move downward in the direction that its altitude decreases most rapidly.

It is \( -\nabla f (1, 1) = (-2a, 2b) \)

\[ \text{Diagram showing the function and its gradient at a point.} \]