I have done
Ex. 5.4  all
Ex. 5.5  11, 15, 28
Ex. 6.1  1, 3, 5, 7
Ex. 6.2  2
Ex. 5.4.

Q11 Evaluate \( \iint_D y^3 (x^2+y^2)^{-\frac{\pi}{2}} \, dx \, dy \), where \( D \) is the region determined by the conditions \( \frac{1}{2} \leq y \leq 1 \) and \( x^2+y^2 \leq 1 \).

\[
\begin{align*}
\text{Pl.} & \quad \iint_D y^3 (x^2+y^2)^{-\frac{\pi}{2}} \, dx \, dy \\
& = \int_{\frac{1}{2}}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^3 (x^2+y^2)^{-\frac{\pi}{2}} \, dx \, dy \\
& = \int_{\frac{\sqrt{3}}{4}}^{\frac{\sqrt{5}}{4}} \int_{\frac{1}{2}}^{\sqrt{1-x^2}} y^3 (x^2+y^2)^{-\frac{\pi}{2}} \, dy \, dx \\
& = \int_{\frac{\sqrt{3}}{4}}^{\frac{\sqrt{5}}{4}} \left[ \int_{\frac{1}{2}}^{\sqrt{1-x^2}} (-y^2) \, d(x^2+y^2)^{-\frac{\pi}{2}} \right] \, dx \\
& \quad \text{integration by parts} \\
& = \int_{\frac{\sqrt{3}}{4}}^{\frac{\sqrt{5}}{4}} (-y^2 (x^2+y^2)^{-\frac{1}{2}} \left|_{\frac{1}{2}}^{\sqrt{1-x^2}} + \int_{\frac{1}{2}}^{\sqrt{1-x^2}} 2y (x^2+y^2)^{-\frac{1}{2}} \, dy \right) \, dx \\
& = \int_{\frac{\sqrt{3}}{4}}^{\frac{\sqrt{5}}{4}} (1-x^2)(x^2+(1-x^2))^{-\frac{1}{2}} - \frac{1}{4}) (x^2 + \frac{1}{4})^{-\frac{1}{4}} \, dx \\
& = 11
\end{align*}
\]
\[
+ \int_{\frac{-1}{N^3}}^{\frac{1}{N^3}} \left( \int_{\frac{-1}{2}}^{\frac{1}{2}} \left( x^2 + y^2 \right)^{-\frac{1}{2}} \, dy \right) \, dx
\]

\[
= \int_{\frac{-1}{N^3}}^{\frac{1}{N^3}} \left[ x^2 - 1 + \frac{1}{4x^2 + \frac{1}{4}} \right] \, dx + \int_{\frac{-1}{N^3}}^{\frac{1}{N^3}} 2 \sqrt{x^2 + y^2} \left[ \sqrt{x^2 + \frac{1}{4}} \right] \, dx
\]

\[
= \int_{\frac{-1}{N^3}}^{\frac{1}{N^3}} \left( x^2 - 1 + \frac{1}{4x^2 + \frac{1}{4}} \right) \, dx + \int_{\frac{-1}{N^3}}^{\frac{1}{N^3}} 2 \sqrt{x^2 + \frac{1}{4}} \, dx
\]

\[
= \int_{\frac{-1}{N^3}}^{\frac{1}{N^3}} x^2 + 1 + \frac{1}{4x^2 + \frac{1}{4}} - 2 \sqrt{x^2 + \frac{1}{4}} \, dx
\]

\[
= \frac{x^3}{3} \left[ \frac{-1}{N^3} \right] + 2 \sqrt{\frac{3}{N^4}} + \int_{\frac{-1}{N^3}}^{\frac{1}{N^3}} \frac{dx}{4x^2 + \frac{1}{4}} - 2 \int_{\frac{-1}{N^3}}^{\frac{1}{N^3}} \sqrt{x^2 + \frac{1}{4}} \, dx - (\star)
\]
Recall the formula sheet at the end of your book.

By 66.

\[-2 \int \frac{\sqrt{3}}{2} \sqrt{x^2 + \frac{1}{4}} \, dx = -2 \left[ \frac{2x}{4} \sqrt{x^2 + \frac{1}{4}} \left| \begin{array}{c} \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \end{array} \right. + \frac{4(4)}{8} \int \frac{1}{\sqrt{4} \sqrt{x^2 + \frac{1}{4}}} \, dx \right] = -2 \left[ \sqrt{\frac{3}{4}} + \frac{1}{8} \int \frac{1}{\sqrt{x^2 + \frac{1}{4}}} \, dx \right] = -2 \sqrt{\frac{3}{4}} + \frac{-1}{4} \int \frac{dx}{\sqrt{x^2 + \frac{1}{4}}} \right] \]

\[\therefore \, (\star) = \frac{2}{3} \left( \frac{3}{4} \right)^{\frac{3}{2}} + 2 \sqrt{\frac{3}{4}} - 2 \sqrt{\frac{3}{4}} = \sqrt{\frac{3}{4}} \]

\[\therefore \, \int_D y^3 (x^2 + y^2)^{-\frac{3}{2}} \, dx \, dy = \sqrt{\frac{3}{4}}. \]

Remark:

There may be a better way to compute the integral. If you find it, please share it with us! I would greatly appreciate it.
Ex. 5.5

All the solid bounded by \( x = y \), \( z = 0 \), \( y = 0 \), \( x = 1 \) and \( x + y + z = 0 \)

Find the volume.

\[
\text{Vol} = \int_0^1 \int_0^x \int_{-x-y}^0 dz \ dy \ dx
\]

\[= \int_0^1 \int_0^x (x+y) \ dy \ dx\]

\[= \int_0^1 \left[ xy + \frac{y^2}{2} \right]_0^x \ dx\]

\[= \int_0^1 x^2 + \frac{x^2}{2} \ dx\]

\[= \frac{3}{2} \int_0^1 x^2 \ dx\]

\[= \frac{3}{2} \left[ \frac{1}{3} x^3 \right]_0^1 = \frac{1}{2}\]

\( \therefore \)

\( Q. \)
\[ Q15 \quad \iiint_W (x^2 + y^2 + z^2) \, dx \, dy \, dz \; ; \; W \text{ is the region bounded by } x+y+z = a \text{ (where } a > 0), \; x=0, \; y=0, \text{ and } z=0. \]

\[ \text{Pf:} \quad \iiint_W (x^2 + y^2 + z^2) \, dx \, dy \, dz \]

\[ = \int_0^a \int_0^{a-x} \int_0^{a-x-y} (x^2 + y^2 + z^2) \, dz \, dy \, dx \]

\[ = \int_0^a \int_0^{a-x} x^2z + y^2z + \frac{z^3}{3} \bigg|_0^{a-x-y} \, dy \, dx \]

\[ = \int_0^a \int_0^{a-x} (y^2 + x^2)(a-x-y) + \frac{(a-x-y)^3}{3} \, dy \, dx \]

\[ = \int_0^a (a-x)^2 \left( a^2 - 2ax + 4x^2 \right) / 6 \, dx \]

\[ = \frac{a^5}{20} \]

\[ \text{P.S.} \]
Q28.
Let \( W \) be the region bounded by the planes \( x=0, \ y=0, \ z=0, \ x+y=1 \) and \( z=x+y \).

(a) Find the volume of \( W \)

\[
\text{Vol} = \int_0^1 \int_0^{1-x} \int_0^{x+y} dz \, dy \, dx
\]

\[
= \int_0^1 \int_0^{1-x} (x+y) \, dy \, dx
\]

\[
= \int_0^1 x(1-x) + \frac{(1-x)^2}{2} \, dx
\]

\[
= \frac{1}{3}
\]

I think you can do part b & c.
Ex. 6.1

Q1 Let \( S^* = (0,1] \times [0, 2\pi) \) and define \( T(r, \theta) = (r \cos \theta, r \sin \theta) \). Determine the image set \( S \). Show that \( T \) is one-to-one & onto on \( S^* \).

**Pf:** Fix \( r \), \( \{ T(r, \theta) \mid \theta \in [0, 2\pi) \} \) is a circle. Since \( S^* \) contains \( 0 < r \leq 1 \).

\( \therefore \) The image set \( S^* = T(S) \) is

\[
\text{It is a unit disc without the origin.}
\]

To show \( T \) is 1-1

\[
T(r_1, \theta_1) = T(r_2, \theta_2)
\]

i.e. \((r_1 \cos \theta_1, r_1 \sin \theta_1) = (r_2 \cos \theta_2, r_2 \sin \theta_2)\)
\[ r_1 \cos \theta_1 = r_2 \cos \theta_2 \quad \text{(i)} \]
\[ r_1 \sin \theta_1 = r_2 \sin \theta_2 \quad \text{(ii)} \]

\[(i)^2 + (ii)^2 \Rightarrow r_1^2 = r_2^2 \Rightarrow r_1 = r_2 \]

From (i), \( \cos \theta_1 = \cos \theta_2 \Rightarrow \theta_1 = \frac{\theta_2}{2} \) or \( 2\pi - \theta_2 \)

From (ii), \( \sin \theta_1 = \sin \theta_2 \Rightarrow \theta_1 = \frac{\theta_2}{2} \) or \( \pi - \theta_2 \)

By (i) & (ii) \( \Rightarrow \theta_1 = \theta_2 \).

\[ T \sim 1 \mathbf{1} \mathbf{1} \]

S. is onto \( S^* \) because \( T(S^*) = S \)
Q3 Let \( D^* = [0, 1] \times [0, 1] \) and define \( T \) on \( D^* \) by \( T(u, v) = (-u^2 + 4u, v) \). Find the image \( D \).

Is \( T \) one to one?

\[
\begin{align*}
\text{pf:} & \quad -u^2 + 4u \text{ maps } u \text{ from } [0, 1] \text{ to } [0, 3] \\
v & \text{ maps } v \text{ from } [0, 1] \text{ to } [0, 1]
\end{align*}
\]

\( \therefore \text{ the image of } D \) is

\[
\begin{array}{c}
\text{To show } T \text{ is } 1-1 \text{ on } D \\
T(u_1, v_1) = T(u_2, v_2) \quad \text{where } (u_1, v_1), (u_2, v_2) \text{ are in } D \\
(-u_1^2 + 4u_1, v_1) = (-u_2^2 + 4u_2, v_2)
\end{array}
\]

\( \therefore -u_1^2 + 4u_1 = -u_2^2 + 4u_2, \quad v_1 = v_2 \)

\( u_2^2 - u_1^2 + 4(u_1 - u_2) = 0 \)

\( (u_2 - u_1)(u_2 + u_1) + 4(u_1 - u_2) = 0 \)

\( (u_2 - u_1)(u_2 + u_1 - 4) = 0 \)

\( \Rightarrow u_2 = u_1 \quad \text{why?} \quad \text{Note } (u_2 + u_1 < 4) \quad \text{as } (u_2, v_2) \text{ are in } D \)
Q5. Let $D^* = [0,1] \times [0,1]$ and define $T$ on $D^*$ by $T(x^*, y^*) = (x^* y^*, x^*)$. Determine the image set $D$. Is $T$ one to one? If not, can we eliminate some subset of $D^*$ so that on the remainder $T$ is one to one?

\[ u = x^* y^* \]
\[ v = x^* \]

\[ T \text{ is not } 1-1 \]
\[ T(0,1) = (0,0) = T(0,0) \]
In order to find a subset of $D^*$ s.t. $T$ is one to one on it.

Let us study $T$.

$$T(x_1, y_1) = T(x_2, y_2)$$

$$\Rightarrow (x_1, y_1, x_1) = (x_2, y_2, x_2)$$

$$\Rightarrow x_1 = x_2 \quad \text{and} \quad x_1 y_1 = x_2 y_2$$

If $x_1$, neither $x_1$ nor $x_2$ are not zero, then $y_1 = y_2, x_1 = x_2$

It is easy to verify that $T$ is one to one on $D^* \setminus \{x = 0\}$

It means "elimination".
Let \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be the spherical coordinate mapping defined by \( (\rho, \phi, \theta) \mapsto (x, y, z) \), where
\[
x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.
\]
Let \( D^* \) be the set of points \( (\rho, \phi, \theta) \) such that \( \phi \in [0, \pi] \), \( \theta \in [0, 2\pi] \), \( \rho \in [0, 1] \). Find \( D = T(D^*) \).
Is \( T \) one to one? If not, can we eliminate some subset of \( D^* \) so that, on the remainder, \( T \) will be one to one?

\[ P_f: \text{We have } x^2 + y^2 + z^2 = \rho^2 \text{ where } 0 \leq \rho \leq 1. \]

Also, fixing \( \rho \), as \( \theta \in [0, 2\pi] \), \( \phi \in [0, \pi] \), from the geometrical point of view, we know the image is a sphere with centre origin and radius \( \rho \).

Now, \( \rho \) ranges from 0 to 1, we have \( D \) is a unit ball.
$T$ is not $(-1)$.

as $T(0, \pi, \frac{\pi}{2}) = T(0, 2\pi, \frac{\pi}{2})$

In order to find a subset $\tilde{D}$ of $D^*$

s.t. $T$ is one to one on $D^* - \tilde{D}$.

Let us study $T$.

$T(\rho_1, \phi_1, \theta_1) = T(\rho_2, \phi_2, \theta_2)$

\[\Rightarrow \rho_1 \cos \phi_1 = \rho_2 \cos \phi_2 \quad - (i)\]

\[\rho_1 \sin \phi_1 \sin \theta_1 = \rho_2 \sin \phi_2 \sin \theta_2 \quad - (ii)\]

\[\rho_1 \sin \phi_1 \cos \theta_1 = \rho_2 \sin \phi_2 \cos \theta_2 \quad - (iii)\]
\[ \Rightarrow \phi_1^2 = \phi_2^2 \quad \text{by} \quad (i)^2 + (ii)^2 + (iii)^2 \]

1. If neither \( \phi_1 \) nor \( \phi_2 \) are zero,
then, we have \( \phi_1 = -\phi_2 + \pi \) or \( \phi_2 \) by (i)
\[ \text{as } 0 < \phi_1, \phi_2 < \pi \]
\[ \therefore \phi_1 = \phi_2 \]

2. If neither \( \sin \phi_1 \) nor \( \sin \phi_2 \) are zero,
by (iii) & (iii), we have \( \Theta_1 = \Theta_2 + 2n\pi \)
where \( n \in \mathbb{N} \)
and \( 0 \leq \Theta_1, \Theta_2 \leq 2\pi \)
\[ \therefore \Theta_1 = \Theta_2 \]

3. If neither \( \Theta_1 \) nor \( \Theta_2 \) are 2\pi
then \( \Theta_1 = \Theta_2 \)

\[ \therefore T \text{ is one to one on} \]
\[ D^* = \{ \{ \rho = 0 \} \cup \{ \theta = 2\pi \} \cup \{ \phi = \pi \frac{3}{2} \} \} \]
Q2. Define

\[ T(x^*, y^*) = \left( \frac{x^* - y^*}{\sqrt{2}}, \frac{x^* + y^*}{\sqrt{2}} \right) \]

Show that \( T \) rotates the unit square, \( D^* = [0, 1] \times [0, 1] \)

**Pf:** \( T \) is a linear map

And \( T(0, 0) = O \), \( T(1, 1) = (0, -\sqrt{2}) \)

\( T(1, 0) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \), \( T(0, 1) = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \)

You can check that \( T(0^*) \) is a square.

Also, \( T \) rotates \( D^* \) anti-clockwise by \( \frac{\pi}{4} \) about the origin \( T(0, 0) \).
Ex. 6.2

Q2 Let \( D \) be the region \( 0 \leq y \leq x \) and \( 0 \leq x \leq 1 \).

Evaluate \( \iint_D (x+y) \, dx \, dy \) by making the change of variables \( x = u + v, \ y = u - v \).

Check your answer by evaluating the integral directly using an iterated integral.

\[
\text{Pf:} \quad \iint_D (x+y) \, dx \, dy = \int_0^1 \int_0^x (x+y) \, dy \, dx
\]

\[
= \int_0^1 xy + \frac{y^2}{2} \bigg|_0^x \, dx
\]

\[
= \int_0^1 x^2 + \frac{x^2}{2} \, dx
\]

\[
= \int_0^1 \frac{3x^2}{2} \, dx
\]

\[
= \frac{x^3}{2} \bigg|_0^1
\]

\[
= \frac{1}{2}
\]
To change variable

let \( T(u, v) = (u+v, u-v) \)

\[\begin{align*}
\text{Reason, since } T & \text{ is linear} \\
\therefore T^{-1}(D) & \text{ is determined} \\
\text{by } T^{-1}(0, 0), T^{-1}(1, 0) \\
\text{and } T^{-1}(1, 1) \text{ where} \\
(0,0), (1,0) \text{ and (1,1)} \\
\text{are vertices of } D
\end{align*}\]

\[\begin{align*}
\int \int_D (x+y) \, dx \, dy & = \int \int_{T^{-1}(D)} (u+v) + (u-v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv \\
& = 2 \int_0^{1/2} \int_0^u 2u \, dv \, du + 2 \int_{1/2}^1 \int_0^{1-u} 2u \, dv \, du \\
& = 4 \int_0^{1/2} u^2 \, du + 4 \int_{1/2}^1 u(1-u) \, du \\
& = 4 \left[ \frac{u^3}{3} \right]_0^{1/2} + 2 \left[ u^2 \right]_{1/2}^1 - 4 \left[ \frac{u^3}{3} \right]_{1/2}^1 \\
& = \frac{1}{2} \left( \frac{1}{3} \right) + (2 - \frac{1}{2}) - \frac{4}{3} + \frac{1}{2} \left( \frac{1}{3} \right) \\
& = \frac{1}{2} + \frac{1}{2} = \frac{1}{2}
\end{align*}\]