RESEARCH STATEMENT

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PAST RESEARCH

The Langlands Program is a cornerstone of modern number theory. A prominent component of the program is the study of automorphic representations, which are believed to contain much arithmetic information. For instance, it is conjectured that there exists a canonical correspondence between automorphic representations of a reductive group $\mathbf{G}$ and homomorphisms from the absolute Galois group to the L-group of $\mathbf{G}$. If proven, the statement provides a synthesis of two major disciplines in mathematics: representation theory, and arithmetic.

Suppose the conjecture is valid. Then, for any reductive group $\mathbf{H}$ whose L-group embeds into that of another reductive group $\mathbf{G}$, the automorphic representations of $\mathbf{H}$ should "lift" to those of $\mathbf{G}$. This conjectural principle is known as Langlands Functoriality. One of the first examples of Langlands Functoriality was given by Jacquet and Langlands in [JL]. They establish there the lifting of the idèle class characters of a quadratic extension E of the base field F to the automorphic representations of $\mathrm{GL}(2)$. Moreover, it is known that such liftings provide a classification of the automorphic representations of $\mathrm{GL}(2)$ which are invariant under tensor product with the class field character of E/F. Other examples of liftings are established in [LL], [L], [K], [AC], [Ro], and [F]. With the exception of [JL], all the works cited above make use of a powerful technique known as the trace formula, which relates the traces of operators defined by the automorphic representations of different groups.

The first part of my Ph.D. thesis gives a classification of the automorphic representations of the symplectic group of similitudes $\mathrm{GSp}(2)$ which are invariant under tensor product with a fixed quadratic idèle class character ε (In which case we say the representation is *ε-invariant*). The classification is expressed in terms of liftings of automorphic representations of two rank one twisted endoscopic groups $\mathbf{H_1}, \mathbf{H_2}$ of $\mathrm{GSp}(2)$. Under the condition that all automorphic representations considered have at least two elliptic local components, we have established the following theorem:

Theorem. *Suppose $\{\Pi\}$ is a global packet of $\mathrm{GSp}(2)$ which is the lift of some automorphic representation/packet of one or both of $\mathbf{H_1}$ and $\mathbf{H_2}$. Then, $\{\Pi\}$ contains at least one representation which is ε-invariant. Moreover, $\{\Pi\}$ is unstable if it is the lift of both twisted endoscopic groups, but the converse is false.*

If $\{\Pi\}$ is a discrete spectrum global packet of $\mathrm{GSp}(2)$ which is not the lift of any discrete spectrum representation/packet of the twisted endoscopic groups, then $\{\Pi\}$ does not contain any discrete spectrum automorphic representation which is ε-invariant.

The classification doubles as an example of Langlands Functoriality, and it may be viewed as the $\mathrm{GSp}(2)$ analogue of the aforecited result in [JL]. The methodology, however, is inspired by [K] and [F]. More precisely, my thesis uses the twisted version [CLL] of Arthur's trace formula [A], [A1], and Kottwitz-Shelstad's stabilization of the trace formula [KS].

The second part of the thesis uses the global results obtained in the first part to deduce a list of equations relating the traces of operators defined by p-adic representations of $\mathrm{GSp}(2)$ and those of the twisted endoscopic groups. It may be viewed as a case study

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of the local component of Langlands Functoriality, which is a conjecture on the lifting of admissible representations of p-adic groups.

Lastly, the following theorem is established in the appendix of the thesis:

Theorem. *In the context of* $\mathrm{GSp}(2)$ *and its twisted endoscopic groups* $\mathbf{H_1}$ *and* $\mathbf{H_2}$*, the Fundamental Lemma holds.*

My results should find interest among researchers who study automorphic representations — in particular of $\mathrm{GSp}(2)$, Siegel modular forms, and harmonic analysis of p-adic groups. Besides being an important problem in its own right, functorial lifting of automorphic representations is known to have great potential establishing classical number theoretic results, see [GJ] for example. It would certainly be interesting to investigate what application my thesis might have in classical number theory.

Odd Degree Cyclic Base change for $\mathrm{U}(3)$**.** Let E/F be a quadratic number (resp. p-adic) field extension, and F' an odd degree cyclic field extension of F. In [CF], we establish a base-change functorial lifting of automorphic (resp. admissible) representations from the unitary group $\mathrm{U}(3, E/F)$ associated with E/F to the unitary group $\mathrm{U}(3, F'E/F')$. As a consequence, we classify the invariant packets of $\mathrm{U}(3, F'E/F')$, namely those which contain (irreducible) automorphic (resp. admissible) representations which are invariant under the action of the Galois group $\mathrm{Gal}(F'E/E)$. To do this we use the trace formula technique, and well-known results on the base-change lifting from $\mathrm{U}(3, E/F)$ to $\mathrm{GL}(3, E)$ and on the base-change lifting for the general linear groups. We also determine the invariance of individual representations, using Howe correspondence. This work is the first study of base change into an algebraic group whose packets are not all singletons, and which does not satisfy the "strong multiplicity one theorem." Novel phenomena are encountered: e.g. there are invariant packets where not every irreducible automorphic (resp. admissible) member is Galois-invariant. We also obtain local twisted character identities with respect to this base-change lifting.

The main local theorem is as follows:

Theorem. *Let* F *be a* p*-adic field. Let* $\{\pi\}$ *be a local packet on* $\mathrm{U}(3, E/F)$*. With the exception of the case where* $[F' : F] = 3$ *and* $\{\pi\}$ *lifts to a representation* $\tilde{\pi}$ *of* $\mathrm{GL}(3, F)$ (*see* [F1]) *which satisfies* $\tilde{\pi} \cong \tilde{\pi} \otimes \varepsilon_{F'/F}$*, where* $\varepsilon_{F'/F}$ *is the character of* $F'^{\times}$ *associated with the field extension* F'/F *via local class field theory, every member of the local packet* $\{\pi'\}$ *of* $\mathrm{U}(3, F'E/F')$ *which is the lift of* $\{\pi\}$ *is* $\mathrm{Gal}(F'E/E')$*-invariant.*

The case where $[F' : F]$ is even is quite different from the odd case. For instance, in the case $[F' : F] = 2$, the images of the L-homomorphisms from the L-groups of the two twisted endoscopic groups of $\mathrm{U}(3, F'E/F')$ to the L-group of $\mathrm{U}(3, F'E/F')$ are disjoint. This results in very different character identities. Both authors have decided not to pursue a complete study of the even case for the time being.

CURRENT/FUTURE RESEARCH

Currently, I am collaborating with W. T. Gan to formulate an analogue of the local Gross-Prasad conjecture [GP] for symplectic groups. I give now a general overview of the project.

The Gross-Prasad Conjecture. Given p-adic representations π of $\mathrm{SO}(n)$ and σ of $\mathrm{SO}(n-1)$, it is known that $\dim \mathrm{Hom}_{\mathrm{SO}(n-1)}(\pi|_{\mathrm{SO}(n-1)}, \sigma)$ is at most one [AGRS]. The Gross-Prasad conjecture says that, given a fixed tempered L-parameter for each of $\mathrm{SO}(n)$ and $\mathrm{SO}(n-1)$, the sum $\sum_{\pi,\sigma} \dim \mathrm{Hom}_{G_{n-1}}(\pi|_{G_{n-1}}, \sigma)$ is equal to one. Here, the sum is over all the representations π and σ, on inner forms G_n and G_{n-1} of $\mathrm{SO}(n)$ and $\mathrm{SO}(n-1)$, which are associated with the given L-parameters.

Formulation of a Conjecture for Symplectic Groups. Towards the goal of eventually formulating a statement regarding symplectic groups analogous to the Gross-Prasad conjecture, we hope to first formulate a parallel statement to [AGRS] for symplectic groups. Instead of examining the space of homomorphisms between representations of $\mathrm{Sp}(2n)$ and $\mathrm{Sp}(2n-2)$, we consider the Fourier-Jacobi model $\mathrm{FJ}_\sigma(\pi)$, associated with a representation σ of $\widetilde{\mathrm{Sp}(2n-2)}$, of a given representation π of $\mathrm{Sp}(2n)$. The main idea is to use local θ-correspondence ([MWV]) to show that the main statement in [AGRS] is equivalent to a statement on $\dim \mathrm{FJ}_\sigma(\pi)$. In the case $n = 2$, M. Baruch and S. Rallis have shown that a representation of $\mathrm{Sp}(4)$ has at most one Fourier-Jacobi model associated with a representation of $\widetilde{\mathrm{SL}(2)}$ (see [BR]). We would like to prove the same statement for general n. The next step is naturally to consider how the dimension of the Fourier-Jacobi model depends on the representations. For this, we can settle the case of $(\mathrm{Sp}(4), \widetilde{\mathrm{SL}(2)})$, thus answering a question of Adler-Prasad [AP] regarding the dimension of the Fourier-Jacobi model for this pair of groups. Another problem which we might consider is to formulate a parallel statement to [AGRS] for representations of $\mathrm{Sp}(2n) \times \widetilde{\mathrm{Sp}(2n-2r)}$, where $r > 1$.

Character Identities for the Local Packets of $\mathrm{GSp}(4)$. Another project is as follows: W. T. Gan and S. Takeda have given a full classification of the p-adic representations of $\mathrm{GSp}(4)$ (see [GT]), and they have in particular defined the local packets of the group, using θ-correspondence. However, there is another way to define local packets, namely via local character identities [LL]. It is not immediately clear what the local character identities are which the packets defined by Gan and Takeda satisfy. Gan and I are hopeful that the problem could be tackled using the trace formula technique.

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