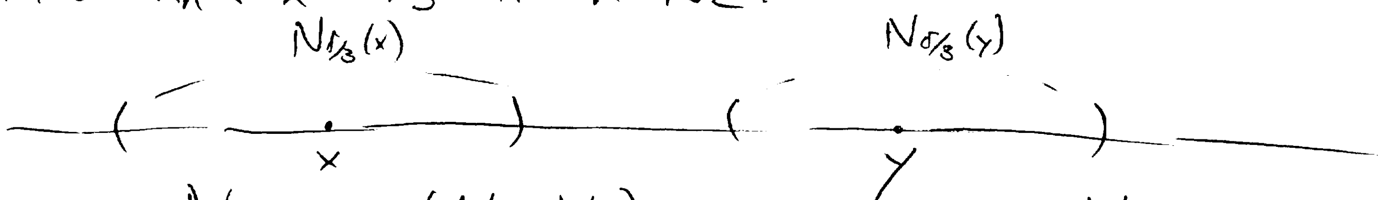
1a) WE SAY THAT $\{x_n\}$ CONVERGES TO x IF FOR ALL $\varepsilon > 0$, THERE EXISTS A $N_\varepsilon \in \mathbb{N}$ FOR WHICH $d(x_n, x) < \varepsilon$ WHENEVER $n > N_\varepsilon$.

b) WITHOUT LOSS OF GENERALITY, ASSUME $S = y$. THEN WE CLAIM THAT $\exists\, N \in \mathbb{N}$ SUCH THAT $S_n = y_n$ FOR ALL $n > N$. (WHY IS THIS CLAIM SUFFICIENT TO PROVE THE STATEMENT?)

LET $\delta := d(y,x) = y - x$. SINCE $y_n \longrightarrow y$, $\exists\, N_1 \in \mathbb{N}$ S.T. $d(y_n, y) < \delta/3$ $\forall\, n > N_1$; THUS $y_n > y - \delta/3$ IF $n > N_1$. SIMILARLY, $\exists\, N_2 \in \mathbb{N}$ S.T. $d(x_n, x) < \delta/3$ $\forall\, n > N_2$; THUS $x_n < x + \delta/3$ IF $n > N_2$.



LET $N := \max(N_1, N_2)$. THEN FOR $n > N$,

$$y_n - x_n > (y - \delta/3) - (x + \delta/3) = \delta - \delta/3 - \delta/3 = \delta/3 > 0.$$

THUS, IF $n > N$, $\max(x_n, y_n) = y_n$.

NOTE: WE HAVE ASSUMED THAT $x \neq y$. IF $x = y$, THE CLAIM CAN BE EASILY PROVEN (THINK ABOUT HOW!)

2) FIX $\varepsilon > 0$. THEN THE COLLECTION

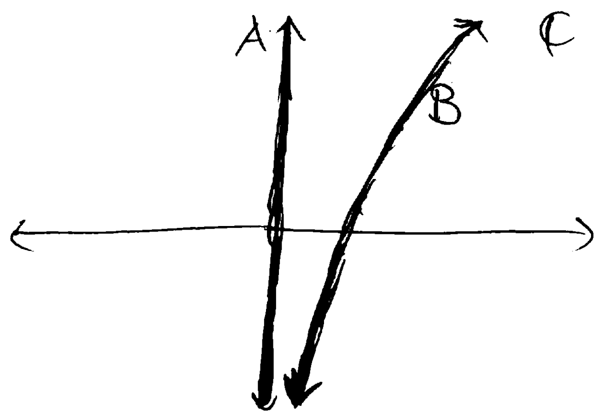
$$\{N_\varepsilon(x) : x \in E\}$$

IS AN OPEN COVER OF E. SINCE E IS COMPACT, THIS COVER ADMITS A FINITE SUBCOVER, WHICH PROVES THE CLAIM.

3a) ANY FINITE SET OF POINTS IN $\mathbb{R}$ WILL SUFFICE. (CHECK THE DETAILS!)

b) DEFINE A AND B AS FOLLOWS:

$$A = \{\alpha i : \alpha \in \mathbb{R}\}$$
$$B = \{e^{\alpha} + \alpha i : \alpha \in \mathbb{R}\}$$



IT IS NOT OBVIOUS THAT THESE SETS ARE CLOSED! YOUR BEST BET TO CHECK THIS FACT WOULD BE TO SHOW THAT THEIR COMPLIMENTS ARE OPEN.

IT IS MORE OBVIOUS THAT $A \cap B = \emptyset$.

DEFINE
$$a_n := -ni \in A$$
$$b_n := e^{-n} - ni \in B.$$

THEN $a_n - b_n = -e^{-n} \xrightarrow{n \to \infty} 0$ (CHECK THIS!).

4a) FALSE. LET $E = \mathbb{Q}$. WE HAVE SHOWN THAT $E' = \mathbb{R}$, WHICH IS UNCOUNTABLE.

b) FALSE. USE PROBLEM 3.16 IN RUDIN (WHICH HAD BEEN ASSIGNED!)

5) $\underline{p \in \overline{E} \Rightarrow \exists \{p_n\} \longrightarrow p}$:

IF $p \in E$, THEN THE SEQUENCE $p_n \equiv p$ IS CLEARLY IN E AND CONVERGES TO p.

IF $p \in E'$, THEN FOR EACH n, THERE EXISTS A $p_n \in E$ CONTAINED IN $B_{1/n}(p)$ BY DEFINITION. THIS SEQUENCE $\{p_n\}$ CONVERGES TO p AS DESIRED.

$\underline{\exists \{p_n\} \longrightarrow p \Rightarrow p \in \overline{E}}$:

EVERY NEIGHBORHOOD OF p CONTAINS ~~FINITELY MANY~~ ALL BUT FINITELY MANY (THUS COUNTABLY MANY) ELEMENTS OF p_n; HENCE, SINCE $\{p_n\} \subset E$, p IS A LIMIT POINT OF E.

6) WE HAVE THAT

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|\sqrt{x_n} - \sqrt{x}|\,|\sqrt{x_n} + \sqrt{x}|}{|\sqrt{x_n} + \sqrt{x}|} = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}}$$

SINCE $x_n, x > 0$, WE HAVE THAT $\sqrt{x_n} + \sqrt{x} > \sqrt{x_n}$, SO

$$\frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} < \frac{|x_n - x|}{\sqrt{x}}.$$

PICK $\varepsilon > 0$. THEN $\exists\, N \in \mathbb{N}$ S.T. $|x_n - x| < (\sqrt{x})\varepsilon \;\forall\, n > N$.

THUS, IF $n > N$,

$$|\sqrt{x_n} - \sqrt{x}| < \frac{|x_n - x|}{\sqrt{x}} < \varepsilon.$$ THEREFORE, $\sqrt{x_n} \longrightarrow \sqrt{x}$.

7) IF EACH A_i IS OPEN, THEN $\bigcup_{i=1}^{\infty} A_i$ IS OPEN AS WELL. YET 0 IS NOT AN INTERIOR POINT OF $[0,1]$, SO $[0,1]$ IS NOT OPEN. THUS, NO SUCH FAMILY $\{A_i\}$ CAN BE FOUND.