

Midterm Solutions

MTH 140B

1. (a) First, we see that $f'(x) = \frac{2}{3} - \frac{4}{3x^3}$, thus f has its only critical point in $[1, \infty)$ at $x = 2^{1/3}$ and is decreasing on $[1, 2^{1/3})$ and increasing on $(2^{1/3}, \infty)$. Furthermore, $f''(x) = 4x^{-4}$, which is strictly positive on $[1, \infty)$, thus, f has a local minimum at $x = 2^{1/3}$. From this, it only suffices to check that

$$f(1) = \frac{4}{3}, f(2^{1/3}) = 2^{1/3}$$

in order to prove the claim.

- (b) By the definition of a contraction along with our results from part (a), it suffices only to check that there exists some $c < 1$ such that $|f(x) - f(y)| \leq c|x - y|$ for any $x, y \geq 1$. We know from our earlier calculation that $f'(x)$ is increasing on our domain; since $f'(1) = -\frac{2}{3}$ and $\lim_{x \rightarrow \infty} f'(x) = \frac{2}{3}$, we know that $|f'(x)| \leq \frac{2}{3}$ for $x \geq 1$. Then for any $x, y \geq 1$, the Mean Value Theorem implies that there exists some $z \in (x, y)$ for which

$$\begin{aligned} |f(x) - f(y)| &= |f'(z)||x - y| \\ &\leq \frac{2}{3}|x - y| \end{aligned}$$

Thus, $c = \frac{2}{3}$ is a sufficient declaration.

- (c) From (b), we have that

$$|x_1 - 2^{1/3}| \leq \frac{2}{3}|1 - 2^{1/3}|$$

Suppose that $|x_j - 2^{1/3}| \leq (\frac{2}{3})^j |1 - 2^{1/3}|$, $j \in 1, 2, \dots, n$. Then by (a) and (b),

$$|x_{n+1} - 2^{1/3}| \leq \frac{2}{3}|x_n - 2^{1/3}| \leq (\frac{2}{3})^{n+1}|1 - 2^{1/3}|$$

Hence, $\alpha = \frac{2}{3}$ and $C = |1 - 2^{1/3}|$ are suitable constants.

2. (a) We will need to use repeated applications of L'Hôpital's Rule;

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - (1 + \frac{x}{2})}{x^2} &= \lim_{x \rightarrow 0} \frac{\frac{1}{2\sqrt{1+x}} - \frac{1}{2}}{2x} \\ &= \lim_{x \rightarrow 0} \frac{1}{2} \left(\frac{-1}{4(1+x)^{\frac{3}{2}}} \right) = \frac{-1}{8}.\end{aligned}$$

(b) One way to approach the problem is to investigate

$$\lim_{n \rightarrow \infty} n \log\left(1 + \frac{1}{n} + \frac{1}{n^2}\right)$$

Recall from Homework 4, #5, we showed a series expansion for $\log(1+t)$, $|t| < 1$. Using this, we can surmise that

$$\begin{aligned}n \log\left(1 + \frac{1}{n} + \frac{1}{n^2}\right) &= n \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \left(\frac{1}{n} + \frac{1}{n^2}\right)^k}{k} \\ &= 1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)\end{aligned}$$

Thus, it follows that $\lim_{n \rightarrow \infty} n \log\left(1 + \frac{1}{n} + \frac{1}{n^2}\right) = 1$;

hence, $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} + \frac{1}{n^2}\right)^n = e$.

3. (a) We can check via the Ratio Test that

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1} n!}{(n+1)! n^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^n n!}{n! n^n} \right| = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Since this limit exists, Rudin Theorem 3.37 implies $\lim_{n \rightarrow \infty} \left(\frac{n^n}{n!}\right)^{\frac{1}{n}} = e$.

Thus, the radius of convergence of this power series is $\frac{1}{e}$.

(b) Note that for sufficiently large n , $1 \leq \sqrt[n]{\log(n)} \leq \sqrt[n]{n}$; thus the Squeeze Theorem implies that $\lim_{n \rightarrow \infty} \sqrt[n]{\log(n)} = 1$ and thus the radius of convergence is 1.