ASSUME C IS PARAMETRIZED BY ARC-LENGTH.

1a) FIRST NOTE THAT BY THE FRENET FORMULAE,

$$C'' = kn$$

WITH $\|n\| = 1$. SINCE C IS A GEODESIC, WE HAVE THAT n, THE NORMAL TO C, IS PARALLEL TO N, THE ~~NORMAL~~ TO T_pS. THUS SINCE $\|N\| = 1$, $N = \pm n$, AND $C'' = \pm kN$.

SINCE C IS A LINE OF CURVATURE, WE ALSO HAVE THAT

$$(*) \quad N' = \lambda C'.$$

FINALLY, WE AGAIN APPEAL TO THE FRENET FORMULAE TO SEE THAT

$$(**) \quad \pm N' = -kC' - \tau b.$$

BY COMPARING $(*)$ WITH $(**)$ AND RECALLING THAT $\{C', N, b\}$ FORM AN ORTHONORMAL BASIS, WE FIND THAT $\tau \equiv 0$; THAT IS, C IS A PLANAR CURVE.

b) WORK BACKWARDS THROUGH (a) TO SEE THAT

$$\pm N' = -kC'$$

WHICH IMPLIES C IS A LINE OF CURVATURE.

c) IF S IS ANY PLANE IN $\mathbb{R}^3$, THEN ANY NON-STRAIGHT CURVE IN S SATISFIES THE REQUIRED PROPERTIES (WHY?)

2) SUPPOSE C IS ASYMPTOTIC AND A GEODESIC. THUS

$$k_g \equiv 0, \quad k_n \equiv 0$$

AND SO

$$k^2 = k_g^2 + k_n^2 = 0$$

HENCE $k \equiv 0$ AND C IS A STRAIGHT LINE SEGMENT.

IF C IS A STRAIGHT LINE SEGMENT THEN

$k = 0$ IMPLIES

$$0 \le k_g^2 + k_n^2 = 0, \text{ WHICH IMPLIES } k_g = k_n = 0.$$

5. ALONG THE MAXIMUM PARALLEL

$$C_1(t) = \left((a+r)\cos\frac{t}{a+r}, (a+r)\sin\frac{t}{a+r}, 0 \right) \quad t \in [0, 2\pi)$$

WE HAVE THAT

$$C_1'' = \frac{-1}{(a+r)^2} C_1$$

AND AT EACH $p \in C_1$, $N_p = \frac{1}{a+r} p$.

THUS C_1'' IS PARALLEL TO N_p, SO C_1 IS A GEODESIC.

BY (1b), C_1 IS A LINE OF CURVATURE

BY (2), C_1 IS NOT ASYMPTOTIC.

SIMILAR ARGUMENTS ALLOW US TO DRAW THE SAME CONCLUSION REGARDING THE MINIMUM PARALLEL

$$C_2(t) = \left((a-r)\cos\frac{t}{a-r}, (a-r)\sin\frac{t}{a-r}, 0 \right) \quad t \in [0, 2\pi)$$

CONSIDER THE UPPER PARALLEL

$$C_3(t) = \left(a\cos\frac{t}{a}, a\sin\frac{t}{a}, r \right) \quad t \in [0, 2\pi)$$

NOTE THAT

$$C_3'' = \left(\frac{-1}{a}\cos\frac{t}{a}, \frac{-1}{a}\sin\frac{t}{a}, 0 \right)$$

AND AT EACH $p \in C_3$, $N_p = (0,0,1)$. HENCE, C_3 IS NOT A GEODESIC. SINCE $N'_p = \vec{0}$, C_3' IS TRIVIALLY AN EIGENVECTOR OF N', SO C_3 IS A LINE OF CURVATURE.

ALSO, $k_n = \langle N, C_3'' \rangle = 0$, SO C_3 IS AN ASYMPTOTIC.