CR Geometry, Mappings into Spheres, and Sums-Of-Squares
Lecture III

Peter Ebenfelt

University of California, San Diego

Available at http://www.math.ucsd.edu/∼pebenfel/.

Outline - Lecture III

1. Transversal CR immersions into hyperquadrics
2. Basic Rigidity Results
3. The CR Second Fundamental Form and Gauss Equation
4. Analysis of the Gauss Equation; an example
5. References

Rigidity in CR Geometry. A basic problem.

- Let $M = M^n_\ell \subset \mathbb{C}^{n+1}$ be a Levi nondegenerate hypersurface of CR dimension $n \geq 1$ and Levi signature $0 \leq \ell \leq n/2$.
- Consider $\mathcal{F}(M, Q^N_{\ell'})$: the space of transversal CR immersions into the hyperquadric $Q^N_{\ell'} \subset \mathbb{C}^{N+1}$,

$$f\colon M \to Q^N_{\ell'}, \quad T^{1,0}_{f(p)} Q^N_{\ell'} + f_* T^{1,0}_p \mathbb{C}^{n+1} = T^{1,0}_{f(p)} \mathbb{C}^{N+1}.$$

- Equivalence relation in $\mathcal{F}(M, Q^N_{\ell'})$: $f_1 \sim f_2 \iff \exists T \in \mathrm{Aut}(Q^N_{\ell'})$ such that $f_2 = T \circ f_1$.

Basic Problem:

Describe the deformation space $\mathcal{F}(M, Q^N_{\ell'})/\sim$.

- "Rigidity" in this context means $\mathcal{F}(M, Q^N_{\ell'})/\sim$ consists of a single point.

A necessary condition on $Q^N_{\ell'}$.

Proposition 1

If there exists a transversal $f\colon M^n_\ell \to Q^N_{\ell'}$, i.e. $\mathcal{F}(M^n_\ell, Q^N_{\ell'}) \neq \emptyset$, then $N \geq n$ and $\ell' \geq \ell$.

Proof. A direct consequence of Levi form invariance. Pick contact forms θ, θ', and local $(1,0)$-vector frames $L_1, \ldots, L_n$, $L'_1, \ldots, L'_N$ for $M = M^n_\ell$ and $Q^N_{\ell'}$, respectively. The Levi forms of M and $Q^N_{\ell'}$ are represented by $n \times n$ and $N \times N$ matrices E and E'; WLOG with signatures $(\ell, n-\ell)$ and $(\ell', N-\ell')$. We have $f^*\theta' = a\theta$ for some real-valued function a on M.

Exercise: f is transversal at $p \in M \iff a(p) \neq 0$.

In the chosen $(1,0)$-vector frames, $(f_*)|_{T^{1,0}M}$ is represented by an $n \times N$ matrix B, and we have by Levi form invariance: $aE = BE'B^*$. The conclusion now easily follows from standard linear algebra. □

Existence of transversal $f\colon M \to Q^N_{\ell'}$. Basic results.

- There are real-analytic Levi nondegenerate hypersurfaces $M = M^n_\ell$ of any signature ℓ and CR dimension n for which $\mathcal{F}(M, Q^N_{\ell'}) = \emptyset$ for all N and ℓ'. "Counting" argument (local) [7]; see also [1].
- If M is real-algebraic, then $\exists N \geq n, \ell' \geq \ell$ such that $\mathcal{F}(M, Q^N_{\ell'}) \neq \emptyset$; in general $\ell' > \ell$. "Diagonalization" argument [11].
- There are compact, real-algebraic, strictly pseudoconvex M such that $\mathcal{F}(M, S^{2N+1}) = \emptyset$ for every N. [8]; recall $S^{2N+1} = Q^N_0$.

A Basic Example

Let $X = X^{n+1} \subset \mathbb{C}^{N+1}$ be complex analytic variety of dimension n with an isolated singularity at $0 \in X$.

- The Milnor links $M_\epsilon := X \cap S^{2N+1}_\epsilon$, for sufficiently small radii $\epsilon > 0$, are compact strictly pseudoconvex hypersurfaces in X that admit (by construction) transversal CR embeddings $f_\epsilon\colon M_\epsilon \to S^{2N+1}$.

CR complexity of $M = M_\ell^n$.

Definition 1

(i) The CR complexity of $M = M_\ell^n$ is given by

$$\mu(M) := \min\{N - n \colon \mathcal{F}(M, Q_\ell^N) \neq \emptyset\}.$$

(ii) More generally, for $k \geq 0$, the CR k-complexity of $M = M_\ell^n$ is given by

$$\mu_k(M) := \min\{N - n \colon \mathcal{F}(M, Q_{\ell+k}^N) \neq \emptyset,\ N \geq 2(\ell + k)\}.$$

- We note that $0 \leq \mu(M) = \mu_0(M) \leq \infty$.
- Assume $\ell + k < N/2$. Then,

$$\mu_{k+1}(M) \leq \mu_k(M) + 1.$$

Basic rigidity results $f\colon M = M^n_\ell \to Q^N_\ell$.

Theorem 1 (see [4, 5])

Assume $\mu(M) < \infty$, and let $f_0 \in \mathcal{F}(M, Q^{N_0}_\ell)$ with $N_0 - n = \mu(M)$. If $f \in \mathcal{F}(M, Q^N_\ell)$ and

$$(N-n) + \mu(M) < n,$$

then $f = T \circ L \circ f_0$ for some $T \in \mathrm{Aut}(Q^N_\ell)$; L denotes the standard linear embedding $z = (z_1, \ldots, z_{n+1}) \mapsto (z, 0)$.

- Applies to $M = Q^n_0 = S^{2n+1}$, $\mu(M) = 0$, $f_0 = Id$. Faran, Webster [6, 12].

Theorem 2 (see [2])

Assume $\mu(M) < \ell$, and let $f_0 \in \mathcal{F}(M, Q^{N_0}_\ell)$ with $N_0 - n = \mu(M)$. If $f \in \mathcal{F}(M, Q^N_\ell)$, then $f = T \circ L \circ f_0$ for some $T \in \mathrm{Aut}(Q^N_\ell)$.

- Super-rigidity. Applies to $M = Q^n_\ell$, $\ell > 0$. Baouendi–Huang [3].

Adapted pseudohermitian frames.

Recall: A choice of contact form θ on $M = M^n_\ell$ defines a pseudohermitian structure. A (local) CR coframe (θ, θ^α) is admissible if $d\theta = ih_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta}$.

Consider a transversal CR map $f\colon M \to \hat M = \hat M^N_{\ell'}$, and fix an admissible CR coframe (θ, θ^α) on M.

An adapted (local) CR coframe on $\hat M$

There exists a CR coframe $(\hat\theta, \hat\theta^A)$ (locally) on $\hat M$ near $f(M)$ such that

$$f^*(\hat\theta, \hat\theta^A) = (\theta, \theta^\alpha, 0).$$

Moreover, if the Levi form $h_{\alpha\bar\beta}$ is diagonal with ± 1 on diagonal, then we can choose $(\hat\theta, \hat\theta^A)$ such that $h_{A\bar B}$ is diagonal with ± 1 on diagonal.

- Side preserving $(h_{A\bar B}) = I'_{\ell', N-\ell'}$ vs. side reversing $(h_{A\bar B}) = I'_{N-\ell', \ell'}$.

The CR Second Fundamental Form (SFF) of $f\colon M^n_\ell \to \hat{M}^N_{\ell'}$.

Fix (locally) a CR coframe (θ, θ^α) on M; $(h_{\alpha\bar\beta}) = I_{\ell,n-\ell}$. Let $(\hat\theta, \hat\theta^A)$ be an adapted CR coframe on $\hat{M}^N_{\ell'}$ near $f(M)$; $(\hat h_{A\bar B}) = I'_{\ell',N-\ell'}$. By pulling back to M

$$d\hat\theta^A = \hat\omega_B{}^A \wedge \hat\theta^B + \hat\tau^A \wedge \hat\theta, \quad \hat\omega_{A\bar B} + \hat\omega_{\bar B A} = 0$$

we deduce

$$\omega_\alpha{}^\beta = \hat\omega_\alpha{}^\beta, \quad \tau^\alpha = \hat\tau^\alpha,$$

and

$$0 = \hat\omega_\beta{}^a \wedge \theta^\beta + \hat\tau^a \wedge \hat\theta; \quad a = N-n+1, \ldots, N.$$

By using $\hat\tau^B = \hat{A}^B{}_{\bar C}\,\hat\theta^{\bar C}$, we conclude that, pulled back to M,

$$\hat\tau^a = 0, \quad \omega_\beta{}^a := \hat\omega_\beta{}^a = \omega_\beta{}^a{}_\alpha \theta^\alpha, \quad \omega_\beta{}^a{}_\alpha = \omega_\alpha{}^a{}_\beta.$$

The SFF of $f\colon M \to \hat{M}$, cont'd.

Definition 2

For $p \in M$, the SFF of $f\colon M \to \hat{M}$ at p,

$$\mathrm{SFF}_p\colon\ T^{1,0}_p M \times T^{1,0}_p M \to T^{1,0}_{f(p)} \widehat{M} / f_*(T^{1,0}_p M),$$

is defined by

$$\mathrm{SFF}_p(L_\alpha, L_\beta) := \omega_\alpha{}^a{}_\beta[\hat{L}_a].$$

We shall view SFF as a tensor $\omega_\alpha{}^a{}_\beta$ on $\mathbb{C}^n \times \mathbb{C}^{N-n} \times \mathbb{C}^n$. The pseudohermitian connections yield covariant differentals

$$\nabla \omega_\alpha{}^a{}_\beta = d\omega_\alpha{}^a{}_\beta - \omega_\mu{}^a{}_\beta \omega_\alpha{}^\mu + \omega_\alpha{}^b{}_\beta \omega_b{}^a - \omega_\alpha{}^a{}_\mu \omega_\beta{}^\mu.$$

We write $\omega_\alpha{}^a{}_{\beta;\gamma}$ to denote the component in the direction θ^γ and define inductively:

$$\nabla \omega_{\gamma_1}{}^a{}_{\gamma_2;\gamma_3\ldots\gamma_j} = d\omega_{\gamma_1}{}^a{}_{\gamma_2;\gamma_3\ldots\gamma_j} + \omega_{\gamma_1}{}^b{}_{\gamma_2;\gamma_3\ldots\gamma_j}\omega_b{}^a - \sum_{l=1}^{j} \omega_{\gamma_1}{}^a{}_{\gamma_2;\gamma_3\ldots\mu\ldots\gamma_j}\omega_{\gamma_l}{}^\mu.$$

Main steps in proof of Thm 1.

Let $f_0 \colon M = M^n_\ell \to Q^{N_0}_\ell$, $f \colon M \to Q^N_\ell$ be transversal immersions, $N_0 = n + \mu(M) \leq N$. Fix CR coframe (θ, θ^α) on M. In adapted CR coframes for f_0 and f, the SFFs are $(\mathring{\omega}_\alpha{}^a{}_\beta)_{a=1}^{N_0-n}$ and $(\omega_\alpha{}^a{}_\beta)_{a=1}^{N-n}$.

1. **Show:** After suitable choice of adapted CR normal frames:

$$\omega_{\gamma_1}{}^a{}_{\gamma_2;\ldots\gamma_{k+2}} = \mathring{\omega}_{\gamma_1}{}^a{}_{\gamma_2;\ldots\gamma_{k+2}}, \quad k = 0, 1, 2, \ldots. \tag{1}$$

   where extra zeros has been added to $\mathring{\omega}_{\gamma_1}{}^a{}_{\gamma_2;\ldots\gamma_{k+2}}$ for $a > N_0 - n$.

2. **Show:** The identities (1) imply, under additional technical assumption (always OK in context of Thm 1), $\exists T \in \mathrm{Aut}(Q^N_\ell)$ such that $f = T \circ L \circ f_0$.

Here, we will focus on Step 1. For Step 2, see [4].

The (CR) Gauss Equation.

Gauss Equation for $f\colon M = M^n_\ell \to Q^N_{\ell'}$

$$
\begin{aligned}
S_{\alpha\bar\beta\mu\bar\nu} = & - h_{a\bar b}\omega_\alpha{}^a{}_\mu \omega_{\bar\beta}{}^{\bar b}{}_{\bar\nu} \\
& + \frac{\omega_\gamma{}^a{}_\alpha\omega^\gamma{}_{a\bar\beta}h_{\mu\bar\nu} + \omega_\gamma{}^a{}_\mu\omega^\gamma{}_{a\bar\beta}h_{\alpha\bar\nu} + \omega_\gamma{}^a{}_\alpha\omega^\gamma{}_{a\bar\nu}h_{\mu\bar\beta} + \omega_\gamma{}^a{}_\mu\omega^\gamma{}_{a\bar\nu}h_{\alpha\bar\beta}}{n+2} \\
& - \frac{\omega_\gamma{}^a{}_\delta\omega^\gamma{}_a{}^\delta}{(n+1)(n+2)}(h_{\alpha\bar\beta}h_{\mu\bar\nu} + h_{\alpha\bar\nu}h_{\mu\bar\beta}).
\end{aligned}
$$

- The "essential" features of this equation can be captured in a simpler, point-wise polynomial identity:

$$
S(\zeta,\bar\zeta) = -\langle \omega(\zeta), \omega(\zeta)\rangle' + A(\zeta,\bar\zeta))\,\langle \zeta,\zeta\rangle\,.
$$

This will be explained in more detail in a few slides.

Sketch of proof of Gauss Equation.

- Inspecting the structure equation for ${\omega_\alpha}^\beta$ on $Q^N_{\ell'}$,

$$d{\omega_\alpha}^\beta = {\omega_\alpha}^A \wedge {\omega_A}^\beta + {\hat{R}_\alpha}{}^\beta{}_{A\bar{B}}\theta^A \wedge \theta^{\bar{B}} + \dots,$$

and pulling back to M, we conclude:

$$R_\alpha{}^\beta{}_{\nu\bar{\mu}} = \hat{R}_\alpha{}^\beta{}_{\nu\bar{\mu}} - {\omega_\alpha}^a{}_\nu \omega^\beta{}_{a\bar{\mu}}. \tag{2}$$

- A symmetric tensor $T_{\alpha\bar{\beta}\nu\bar{\mu}}$ can be uniquely decomposed as

$$T_{\alpha\bar{\beta}\nu\bar{\mu}} = T^0_{\alpha\bar{\beta}\nu\bar{\mu}} + (h_{\alpha\bar{\beta}}) \otimes (T^1_{\nu\bar{\mu}}),$$

where $T^0_{\alpha\bar{\beta}\nu\bar{\mu}}$ is the trace-free part of $T_{\alpha\bar{\beta}\nu\bar{\mu}}$, i.e., $T^{0\alpha}_{\alpha}{}_{\nu\bar{\mu}} = 0$.

- Taking the trace-free part of (2) and using $\hat{S}_{A\bar{B}C\bar{D}} = 0$ on $Q^N_{\ell'}$ yields Gauss Equation. □

Gauss Equation for sectional curvature.

- Define polynomials in $\zeta = (\zeta^1, \ldots, \zeta^n)$ and $\bar{\zeta}$:

$$S(\zeta, \bar{\zeta}) := S_{\alpha\bar{\beta}\nu\bar{\mu}} \zeta^\alpha \zeta^\nu \overline{\zeta^\beta \zeta^\mu}, \quad \omega^a(\zeta) := \omega_\alpha{}^a{}_\nu \zeta^\alpha \zeta^\nu,$$

and Hermitian forms

$$\langle \zeta, \zeta \rangle := h_{\alpha\bar{\beta}} \zeta^\alpha \overline{\zeta^\beta}, \quad \langle \tau, \tau \rangle' := h_{a\bar{b}} \tau^a \overline{\tau^b}.$$

- $\langle \cdot, \cdot \rangle$ has rank n and signature ℓ,
- $\langle \cdot, \cdot \rangle'$ has rank $N - n$ and signature either $\ell' - \ell$ (side preserving) or $N - \ell' - \ell$ (side reversing). Note: side reversing requires $\ell' + \ell \geq n$.

Gauss Equation

$$S(\zeta, \bar{\zeta}) = -\langle \omega(\zeta), \omega(\zeta) \rangle' + A(\zeta, \bar{\zeta}) \langle \zeta, \zeta \rangle .$$

Analysis of the Gauss Equation. An example (Thm 1).

- Let $f_0\colon M = M^n_\ell \to Q^{N_0}_\ell$, $f\colon M \to Q^N_\ell$ be transversal immersions, $N_0 = n + \mu(M) \le N$. Fix CR coframe (θ, θ^α) on M. In adapted CR coframes for f_0 and f, the SFFs are $(\mathring{\omega}_\alpha{}^a{}_\beta)_{a=1}^{N_0-n}$ and $(\omega_\alpha{}^a{}_\beta)_{a=1}^{N-n}$.
- Recall, $\ell' = \ell \implies (\mathring{h}_{a\bar b}) = I_{(N_0-n)\times(N_0-n)}$, $(h_{a\bar b}) = I_{(N-n)\times(N-n)}$.
- Gauss Equations become:

$$
\begin{aligned}
S(\zeta,\bar\zeta) &= -\|\mathring{\omega}(\zeta)\|^2 + \mathring{A}(\zeta,\bar\zeta)\left\langle \zeta,\zeta\right\rangle,\\
S(\zeta,\bar\zeta) &= -\|\omega(\zeta)\|^2 + A(\zeta,\bar\zeta)\left\langle \zeta,\zeta\right\rangle;
\end{aligned}
$$

- $\|\mathring{\omega}(\zeta)\|^2$ has $\mu(M) = N_0 - n$ terms and $\|\omega(\zeta)\|^2$ has $N - n$ terms.
- Subtracting the two Gauss equations, we obtain:

$$
-\|\mathring{\omega}(\zeta)\|^2 + \|\omega(\zeta)\|^2 = B(\zeta,\bar\zeta)\left\langle \zeta,\zeta\right\rangle. \tag{3}
$$

Huang's Lemma.

Huang's Lemma (see [9])

Let $1 \leq k < n$; $g_1, \ldots, g_k$, $f_1, \ldots, f_k$ holomorphic functions near $0 \in \mathbb{C}^n$ such that

$$\sum_{i=1}^{k} g_i(\zeta)\overline{f_i(\zeta)} = B(\zeta, \bar{\zeta}) \langle \zeta, \zeta \rangle .$$

Then $B(\zeta, \bar{\zeta}) \equiv 0$.

- Applying Huang's Lemma to subtracted Gauss Equations (3) $\implies$

$$\|\mathring{\omega}(\zeta)\|^2 = \|\omega(\zeta)\|^2.$$

- Linear Algebra $\implies$ $\exists U \in U(\mathbb{C}^{N-n})$:

$$\omega(\zeta) = (\mathring{\omega}(\zeta), 0)U. \tag{4}$$

Changing adapted normal frame.

- We write $U = (U_a{}^b)$, where $a, b = 1, \ldots, N - n$. (4) $\implies$

$$\omega_\alpha{}^a{}_\beta = U_b{}^a \mathring{\omega}_\alpha{}^b{}_\beta,$$

where we have added extra zeros to $\mathring{\omega}_\alpha{}^b{}_\beta$ for $b > N_0 - n$.

- We change the adapted CR frame for f by $L_b \mapsto U_b{}^a L_a \implies$

$$\omega_\alpha{}^a{}_\beta = \mathring{\omega}_\alpha{}^a{}_\beta. \tag{5}$$

- We note freedom in choice of U due to zeros in $\mathring{\omega}_\alpha{}^b{}_\beta$. May make further changes in L_a for $a > N_0 - n$ without altering (5).

Conformally Flat Tensors (CFT).

- A tensor $T_{\alpha_1\ldots\alpha_k\bar\beta_1\ldots\bar\beta_l}{}^{a_1\cdots a_m}$ is *conformally flat* (a CFT) if it is of the form $(h_{\alpha\bar\beta})\otimes(T'_{\alpha_1\ldots\alpha_{k-1}\bar\beta_1\ldots\bar\beta_{l-1}}{}^{a_1\ldots a_m})$, i.e., a sum of terms each having a factor $h_{\alpha_i\bar\beta_j}$;

$$T^{a_1\ldots a_m}(\zeta,\bar\zeta)=T_{\alpha_1\ldots\alpha_k\bar\beta_1\ldots\bar\beta_l}{}^{a_1\ldots a_m}\zeta^{\alpha_1}\ldots\zeta^{\alpha_k}\overline{\zeta^{\beta_1}\ldots\zeta^{\beta_\ell}}=A(\zeta,\bar\zeta)\,\langle\zeta,\zeta\rangle\,.$$

- The Gauss Equation can be written

$$S_{\alpha\bar\beta\nu\bar\mu}=-h_{a\bar b}\omega_\alpha{}^a{}_\nu\omega_{\bar\beta}{}^{\bar b}{}_{\bar\mu}\quad \mathrm{mod}\quad CFT. \tag{6}$$

- Since $h_{\alpha\bar\beta}$ is parallel, i.e. $\nabla h_{\alpha\bar\beta}=0$, any covariant derivative of a conformally flat tensor is again conformally flat.

Lemma 1 (see [4], [2])

The covariant derivative tensor $\omega_\alpha{}^a{}_{\beta;\bar\gamma}$ is conformally flat (a CFT).

Completing Step 1 in proof of Thm 1.

- Taking repeated covariant derivatives of Gauss Equation (6), using Lemma 1, yields

$$\begin{aligned} S_{\gamma_1\bar\beta\gamma_2\bar\mu;\gamma_3\ldots\gamma_k} &= -h_{a\bar b}\mathring{\omega}_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3\ldots\gamma_k}\mathring{\omega}_{\bar\beta}{}^{\bar b}{}_{\bar\mu} \mod CFT,\\ S_{\gamma_1\bar\beta\gamma_2\bar\mu;\gamma_3\ldots\gamma_k} &= -h_{a\bar b}\omega_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3\ldots\gamma_k}\omega_{\bar\beta}{}^{\bar b}{}_{\bar\mu} \mod CFT. \end{aligned} \tag{7}$$

- Multiply both equations by $\zeta^{\gamma_1}\ldots\zeta^{\gamma_k}\overline{\zeta^{\beta}\zeta^{\mu}}$, introduce

$$\Omega^a_{(k)}(\zeta) := \omega_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3\ldots\gamma_k}\zeta^{\gamma_1}\ldots\zeta^{\gamma_k}, \quad \Omega^a_{(2)} = \omega^a(\zeta),$$

and subtract the two equations in (7) $\implies$

$$\sum_{a=1}^{N_0-n}\left(\Omega^a_{(k)}(\zeta)-\mathring{\Omega}^a_{(k)}(\zeta)\right)\overline{\omega^a(\zeta)} = \tilde B(\zeta,\bar\zeta)\,\langle\zeta,\zeta\rangle\,. \tag{8}$$

Completing Step 1 in proof of Thm 1; cont'd.

- Applying Huang's Lemma to (8), we conclude

$$\sum_{a=1}^{N_0-n}\left(\Omega^a_{(k)}(\zeta)-\mathring{\Omega}^a_{(k)}(\zeta)\right)\overline{\omega^a(\zeta)}=0,$$

which in tensor form becomes

$$\sum_{a=1}^{N_0-n}\left(\omega_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3\dots\gamma_k}-\mathring{\omega}_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3\dots\gamma_k}\right)\omega_{\bar\beta}{}^{\bar a}{}_{\bar\mu}=0. \qquad (9)$$

- Consider the normal vectors $V_{\gamma_1\dots\gamma_k}:=(\omega_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3\dots\gamma_k})_{a=1}^{N-n}\in\mathbb{C}^{N-n}$, and subspaces (essentially introduced by Lamel [10])

$$E_k:=\operatorname{span}\{V_{\gamma_1\dots\gamma_l}\colon 2\le l\le k\}\subset\mathbb{C}^{N-n}.$$

Completion of Step 1 in special case.

- We note that (9) means that the projection of

$$\omega_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3\ldots\gamma_k} - \mathring{\omega}_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3\ldots\gamma_k}$$

  on $E_2 = \mathring{E}_2$ vanishes.
- If we assume that $\mathring{E}_2 = \mathbb{C}^{N-n}$ (which can only happen if $N = N_0$), then we have completed Step 1:

$$\omega_{\gamma_1}{}^{a}_{\gamma_2;\ldots\gamma_{k+2}} = \mathring{\omega}_{\gamma_1}{}^{a}_{\gamma_2;\ldots\gamma_{k+2}}, \quad k = 2, 3 \ldots .$$

- The proof of Thm 1 is now completed by completing Step 2.

What to do if $\mathring{E}_2 \neq \mathbb{C}^{N-n}$?

- One must then also differentiate the Gauss Equations in the $\theta^{\bar\gamma}$ directions, and use a commutation formula for covariant derivatives [2].
- Doing so, and repeatedly using Huang's Lemma as above, one may deduce that $E_l = \mathring{E}_l$ and the projection of

$$\omega_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3\cdots\gamma_k} - \mathring{\omega}_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3\cdots\gamma_k}$$

  on $\mathring{E}_l$, for any l, vanishes, after possibly additional changes of adapted normal frame. This completes Step I.
- If $\mathring{E}_l = \mathbb{C}^{N-n}$ for some l, then Step 2 can be executed.
- If the $E_l = \mathring{E}_l$ stabilize at $E_\infty \subsetneq \mathbb{C}^{N-n}$, then need to show $f(M)$ is contained in an affine plane section of Q^N_ℓ of appropriate dimension. In the context of Thm 1, this is OK. More on this in next lecture...

The End

M. S. Baouendi, Peter Ebenfelt, and Linda Preiss Rothschild.
Local geometric properties of real submanifolds in complex space.
Bull. Amer. Math. Soc., 37(3):309–336, 2000.

M. Salah Baouendi, Peter Ebenfelt, and Xiaojun Huang.
Super-rigidity for CR embeddings of real hypersurfaces into hyperquadrics.
Adv. Math., 219(5):1427–1445, 2008.

M. Salah Baouendi and XiaoJun Huang.
Super-rigidity for holomorphic mappings between hyperquadrics with positive signature.
J. Differential Geom., 69:379–398, 2005.

Peter Ebenfelt, Xiaojun Huang, and Dmitry Zaitsev.
Rigidity of CR-immersions into spheres.
Comm. Anal. Geom., 12(3):631–670, 2004.

Peter Ebenfelt, Xiaojun Huang, and Dmitry Zaitsev.
The equivalence problem and rigidity for hypersurfaces embedded into hyperquadrics.

Amer. J. Math., 127(1):169–191, 2005.

James J. Faran.
The linearity of proper holomorphic maps between balls in the low codimension case.
J. Differential Geom., 24(1):15–17, 1986.

Franc Forstnerič.
Embedding strictly pseudoconvex domains into balls.
Trans. Amer. Math. Soc., 295(1):347–368, 1986.

X. Huang and M. Xiao.
Chern-moser-weyl tensor and embeddings into hyperquadrics.
preprint; https://arxiv.org/abs/1606.09145, 2012.

Xiaojun Huang.
On a linearity problem for proper holomorphic maps between balls in complex spaces of different dimensions.
J. Differential Geom., 51:13–33, 1999.

Bernhard Lamel.

Holomorphic maps of real submanifolds in complex spaces of different dimensions.
Pacific J. Math, 201(2):357–387, 2001.

S. M. Webster.
Some birational invariants for algebraic real hypersurfaces.
Duke Math. J., 45(1):39–46, 1978.

S. M. Webster.
The rigidity of C-R hypersurfaces in a sphere.
Indiana Univ. Math. J., 28(3):405–416, 1979.