CR Geometry, Mappings into Spheres, and Sums-Of-Squares
Lecture II

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Outline - Lecture II

1. Levi Nondegenerate Hypersurfaces in $\mathbb{C}^{n+1}$
2. Chern-Moser Normal Form
3. E. Cartan's Approach to CR Geometry
4. Pseudohermitian Geometry
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Levi nondegenerate hypersurfaces in $\mathbb{C}^{n+1}$.

Let $M \subset \mathbb{C}^{n+1}$ be a real hypersurface, and $p \in M$.

Definitions.

- M is Levi nondegenerate at p if the Levi form

$$\mathcal{L}_p^\theta\colon\, T_p^{1,0}M \times T_p^{1,0}M \to \mathbb{C}$$

  at p is nondegenerate for some (and hence all) contact forms θ.
- M is strictly pseudoconvex at p if $\mathcal{L}_p^\theta$ is (positive) definite.

Fix $p \in M$. Choose local coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ such that

$$p = (0,0), \quad T_0^{0,1}M = \{w = 0\}, \quad T_0M = \{\operatorname{Im} w = 0\}.$$

Express M in graph form:

$$\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w), \quad \phi(0) = 0,\ d\phi(0) = 0;\ \phi \in C^\kappa.$$

The Levi form.

A computation (see Lecture I) shows that the Levi form $\mathcal{L}_0^\theta$, with $\theta = i\partial\rho|_M$, is represented by

$$\mathcal{L}_0(a,\bar{a}) = \sum_{j,k=1}^{n} \frac{\partial\phi}{\partial z_j \partial \bar{z}_k}(0) a_j \bar{a}_k, \quad a \in T_0^{1,0}M \cong \mathbb{C}^n.$$

Assume: M is Levi nondegenerate at 0; i.e.,

$$\det(\phi_{z_j\bar{z}_k}(0))_{j,k=1}^n \neq 0.$$

A linear change $(z,w) \mapsto (Az, \pm w)$, $A \in GL(\mathbb{C}^n)$, will make

$$(\phi_{z_j\bar{z}_k}(0))_{j,k=1}^n = I_\ell,$$

where $I_\ell =$ diagonal matrix $D(-1,\ldots,-1,+1,\ldots,+1)$, with ℓ "-1" and $n-\ell$ "+1" for some $0 \le \ell \le n/2$. ℓ is called signature of M.

The quadric Q_ℓ^n and weights.

A polynomial change $(z, w) \mapsto (z, w - p(z))$, with $p(z)$ suitable quadratic polynomial, yields

$$\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w) = \langle z, \bar{z} \rangle_\ell + O_{wt}(3), \tag{1}$$

where

$$\langle z, \zeta \rangle_\ell := -\sum_{j=1}^{\ell} z_j \zeta_j + \sum_{j=\ell+1}^{n} z_j \zeta_j$$

and we assign weights $\operatorname{wt} z = 1$, $\operatorname{wt} w = 2$. The quadric Q_ℓ^n is the model

$$\operatorname{Im} w = \langle z, \bar{z} \rangle_\ell.$$

Automorphisms of the model Q^n_ℓ.

The stability group $\mathrm{Aut}_0(Q^n_\ell)$ consists of:

$$(z,w) \mapsto \left(\frac{\lambda(z-aw)U}{1-2izI_\ell a^* - (r+iaI_\ell a^*)w}, \frac{\sigma\lambda^2 w}{1-2izI_\ell a^* - (r+iaI_\ell a^*)w} \right),$$

where $\lambda > 0$, $a \in \mathbb{C}^n$, $r \in \mathbb{R}$, $\sigma = \pm 1$, and

$$U^* I_\ell U = \sigma I_\ell.$$

Proposition 1

Any biholomorphism $\Phi(z,w)$, with $\Phi(0) = 0$ and preserving the form (1) of M, factors uniquely as $\Phi = H \circ \Phi_0$, with $\Phi_0 \in \mathrm{Aut}_0(Q^n_\ell)$ and

$$H(z,w) = (z + f(z,w), w + g(z,w)),$$

where

$$(f(0),\ df(0),\ g(0),\ dg(0),\ g_{z_j z_k}(0),\ \mathrm{Re}\, g_{w^2}(0)) = 0. \qquad (2)$$

Decomposition of power series by type.

Let $F(z,\bar{z},s)$ be a formal power series. F is said to be of type (k,l) if

$$F(rz,t\bar{z},s)=r^k t^l F(z,\bar{z},s),$$

and is then a polynomial in z and $\bar{z}$. Any $F(z,\bar{z},s)$ can be decomposed into type as

$$F(z,\bar{z},s)=\sum_{k,l\geq 0} F_{kl}(z,\bar{z},s),$$

where $F_{kl}(z,\bar{z},s)$ has type (k,l). $F(z,\bar{z},s)$ is Hermitian (real) if

$$F_{lk}(z,\bar{z},s)=\overline{F_{kl}(z,\bar{z},s)}.$$

The trace operator Tr .

If $F_{kl}(z,\bar{z},s)$ has type (k,l), then it has "tensor form"

$$F_{kl}(z,\bar{z},s) = a_{\alpha_1\ldots\alpha_k,\bar{\beta}_1\ldots\bar{\beta}_l}(s) z^{\alpha_1}\ldots z^{\alpha_k}\overline{z^{\beta_1}}\ldots\overline{z^{\beta_l}},$$

where $z=(z^1,\ldots,z^n)$, $\quad \alpha_i,\beta_j = 1,\ldots,n$. We shall write

$$\langle z,\bar{z}\rangle_\ell = h_{\alpha\bar{\beta}} z^\alpha \overline{z^\beta}.$$

The trace of $F_{kl}(z,\bar{z},s)$ is of type $(k-1,l-1)$, defined by

$$\operatorname{Tr} F_{kl}(z,\bar{z},s) = b_{\alpha_1\ldots\alpha_{k-1},\bar{\beta}_1\ldots\bar{\beta}_{l-1}} z^{\alpha_1}\ldots z^{\alpha_{k-1}}\overline{z^{\beta_1}}\ldots\overline{z^{\beta_{l-1}}},$$

where

$$b_{\alpha_1\ldots\alpha_{k-1},\bar{\beta}_1\ldots\bar{\beta}_{l-1}} = h^{\gamma\bar{\mu}} a_{\alpha_1\ldots\alpha_{k-1}\gamma,\bar{\beta}_1\ldots\bar{\beta}_{l-1}\bar{\mu}}, \quad h^{\alpha\bar{\mu}} h_{\beta\bar{\mu}} = \delta^\alpha{}_\beta.$$

Chern-Moser normal form [3].

Theorem CM-1

Let M be given by (1). Then, there is a unique formal transformation of the form

$$(z, w) \mapsto (z + f(z, w), w + g(z, w)),$$

where f, g satisfy the normalization (2), such that M is given by

$$\operatorname{Im} w = \langle z, \bar{z} \rangle_\ell + N(z, \bar{z}, \operatorname{Re} w), \tag{3}$$

where $N(z, \bar{z}, s)$ is in Chern-Moser normal form:

$$\begin{aligned} N_{kl}(z, \bar{z}, s) = 0, \quad \min(k, l) \leq 1; \\ \operatorname{Tr} N_{22}(z, \bar{z}, s) = (\operatorname{Tr})^2 N_{32}(z, \bar{z}, s) = (\operatorname{Tr})^3 N_{33}(z, \bar{z}, s) = 0. \end{aligned} \tag{4}$$

Remark. For a given M, the space $\mathrm{Aut}_0(Q^n_\ell)$ acts on the space of CM normal forms by Proposition 1.

Real-analytic hypersurfaces and geometry.

Theorem CM-2

If M is C^ω, then the unique transformation to normal form in Theorem CM-1 is convergent, i.e., a biholomorphism.

- The first set of equations in (4) corresponds to transforming a given framed, transverse curve (γ, e_α) into

$$(\gamma(t), e_\alpha(t)) = ((0, t), \partial/\partial z^\alpha).$$

- The second set is a system of ODEs (of order 3) for the framed curve. The initial data consist of a direction for γ at 0, an orthonormal basis $\{e_\alpha\}$ at 0 for $T_0^{1,0}M$, and a real parameter fixing the parameterization; these initial conditions are parametrized by $\mathrm{Aut}_0(Q^n_\ell)$.
- The curves γ that yield solutions to this system of ODEs are called chains. These are important geometric objects associated with M.

The CR curvature $S = (S_{\alpha\bar{\beta}\nu\bar{\mu}})$.

The Levi form provides a first, very rough classification of Levi nondegenerate hypersurfaces $M \subset \mathbb{C}^{n+1}$ via the signature ℓ. The next interesting invariant is the CR curvature, defined as follows:

Definition. If M is given at $p \in M$ in normal form (3) and (4), then the CR curvature of M at p is $S_{\alpha\bar{\beta}\nu\bar{\mu}}$, where $N_{22}(z, \bar{z}, 0)$ is given in tensor form:

$$N_{22}(z, \bar{z}, 0) = S_{\alpha\bar{\beta}\nu\bar{\mu}} z^{\alpha} z^{\nu} \overline{z^{\beta} z^{\mu}}. \tag{5}$$

Remarks. Recall that $\mathrm{Tr}\, N_{22} = 0 \implies S_{\alpha\bar{\beta}\nu}{}^{\nu} := h^{\nu\bar{\mu}} S_{\alpha\bar{\beta}\nu\bar{\mu}} = 0$. For $n = 1$ (i.e., in $\mathbb{C}^2$), this means $S_{\alpha\bar{\beta}\nu\bar{\mu}} = 0$, so CR curvature is only interesting when $n \geq 2$. In $\mathbb{C}^2$, the interesting invariant is E. Cartan's "6th order tensor".

- For $n \geq 2$, M is locally "spherical" (equivalent to quadric) $\iff$ $S_{\alpha\bar{\beta}\nu\bar{\mu}} \equiv 0$.

E. Cartan's approach

CR coframes on a CR manifold (hypersurface type).

Let M be a $2n+1$-dimensional CR manifold;

- CR bundle $T^{0,1}M$, CR-dim $M = n$.

In an open subset $U \subset M$:

- Fix a contact form θ on M; $\iff$ θ is real and

$$\theta^{\perp} = T^{1,0}M \oplus T^{0,1}M.$$

- Add linearly independent 1-forms $\theta^1, \ldots, \theta^n$ such that

$$(\theta, \theta^1, \ldots \theta^n)^{\perp} = T^{0,1}M.$$

- Set $\theta^{\bar\alpha} = \overline{\theta^\alpha}$; Convention: $\alpha, \beta, \ldots = 1, \ldots, n$.
- $(\theta, \theta^\alpha, \theta^{\bar\beta})$ is coframe for M in U; (θ, θ^α) is called a CR coframe.

Change of coframe and CTCM coframes.

Any other CR coframe $(\tilde{\theta}, \tilde{\theta}^\alpha)$ in $U \subset M$ must be of the form

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\theta}^\alpha \end{pmatrix} = \begin{pmatrix} u & 0 \\ u^\alpha & {u_\beta}^\alpha \end{pmatrix} \begin{pmatrix} \theta \\ \theta^\beta \end{pmatrix}.$$

For a choice of CR coframe (θ, θ^α),

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} + \theta \wedge \phi_0, \tag{6}$$

where $h_{\alpha\bar{\beta}}$ is the Levi form $\mathcal{L}^\theta(L_\alpha, L_\beta)$ and ϕ_0 a real 1-form, determined only up to $\phi_0 \mapsto \phi_0 + v\theta$.

Definition. A choice of $(\theta, \theta^\alpha, \theta^{\bar{\beta}}, \phi_0)$ (as above) is called a CTCM coframe.

CTCM = Cartan-Tanaka-Chern-Moser, [1, 2, 4, 3].

First prolongation; the bundle of contact forms $E \to M$.

Let $E \to M$ be the $\mathbb{R}_+$ bundle of contact forms such that the Levi form $h_{\alpha\bar\beta}$ has $\ell \leq n/2$ negative eigenvalues. For a fixed such θ and $x \in M$,

$$E_x = \{\omega = u\theta \colon u \in \mathbb{R}_+\}.$$

By (6), we have

$$d\omega = iuh_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta} + \omega \wedge \left(\frac{du}{u} + \phi_0\right),$$

which can be written

$$d\omega = ig_{\alpha\bar\beta}\omega^\alpha \wedge \omega^{\bar\beta} + \omega \wedge \phi, \tag{7}$$

where $g_{\alpha\bar\beta}$ is a constant matrix and $(\omega, \omega^\alpha, \omega^{\bar\beta}, \phi)$ is a coframe on E.

The bundle of CTCM coframes $Y \to E \to M$.

The coframe $(\omega, \omega^\alpha, \omega^{\bar\beta}, \phi)$ on E is determined up to

$$\begin{pmatrix} \tilde\omega \\ \tilde\omega^\alpha \\ \tilde\omega^{\bar\beta} \\ \tilde\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v^\alpha & {v_\nu}^\alpha & 0 & 0 \\ v^{\bar\beta} & 0 & {v_{\bar\mu}}^{\bar\beta} & 0 \\ s & ig_{\gamma\bar\rho}{v_\nu}^\gamma v^{\bar\rho} & -ig_{\gamma\bar\rho}{v_{\bar\mu}}^{\bar\rho} v^\gamma & 1 \end{pmatrix} \begin{pmatrix} \omega \\ \omega^\nu \\ \omega^{\bar\mu} \\ \phi \end{pmatrix}, \tag{8}$$

where $g_{\alpha\bar\beta} = g_{\nu\bar\mu}{v_\alpha}^\nu {v_{\bar\beta}}^{\bar\mu}$.

- Let $Y \to E$ be the bundle of all CTCM coframes, i.e., the bundle of all coframes of the form (8).
- G = group of all matrices in (8) acts on $Y \to E$.
- $Y \to E$ is a principal G-bundle. G-structure on M.

Reduction of $Y \to E \to M$ to a $\{e\}$-structure [3].

Theorem CM-2

There exists a uniquely determined coframe

$$(\omega, \omega^\alpha, \omega^{\bar\beta}, \phi, \phi^\alpha, \phi^{\bar\beta}, \phi_\gamma{}^\alpha, \phi_{\bar\mu}{}^{\bar\beta}, \psi) \tag{9}$$

on Y that satisfy **structure equations** (including (7)). The coframe can be assembled into a Cartan connection on $Y \to E \to M$.

- One of the structure equations has the form

$$d\phi_\gamma{}^\alpha = \phi_\gamma{}^\nu \wedge \phi_\nu{}^\alpha + S_\gamma{}^\alpha{}_{\nu\bar\mu}\omega^\nu \wedge \omega^{\bar\mu} + \dots. \tag{10}$$

- Given a CTCM coframe $(\theta, \theta^\alpha, \theta^{\bar\beta}, \phi)$ ($\implies$ section of Y), the forms (9) can be pulled back to M, and $S_{\alpha\bar\beta\nu\bar\mu} = g_{\gamma\bar\beta}S_\gamma{}^\alpha{}_{\nu\bar\mu}$ yields the CR curvature tensor on M previously defined.

Locally "spherical" CR structures.

Remark. Bianchi identities can be used to show that if

$$S_{\alpha\bar\beta\nu\bar\mu} \equiv 0, \quad \text{on } \pi^{-1}(U) \subset Y$$

for some $U \subset M$, then the coframe (9) on $\pi\colon Y \to M$ (locally over U) coincides with (satisfies the same structure equations as) that of the hyperquadric $\pi\colon Y_0 \to Q^n_\ell$. According to E. Cartan's solution to his "equivalence problem", it follows that there is a diffeomorphism $Y \cong Y_0$ (locally). This pushes down to a CR equivalence $U \cong U' \subset Q^n_\ell$.

- Thus, $S_{\alpha\bar\beta\nu\bar\mu} \equiv 0$ characterizes the hyperquadric locally.

S. Webster and N. Tanaka's approach. Pseudohermitian geometry.

Pseudohermitian geometry and admissible frames

Fix a contact form θ on M. $(M, \mathcal{V} = T^{0,1}M, \theta)$ is called a pseudohermitian manifold. Let (θ, θ^α) be a CR coframe. By a change

$$\theta^\alpha \mapsto \theta^\alpha + u^\alpha \theta,$$

it follows from (6) that we can achieve

$$d\theta = i h_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}}. \tag{11}$$

A CR coframe (θ, θ^α) satisfying (11) is called admissible. The forms θ^α are determined up to changes

$$\theta^\alpha \mapsto u_\nu{}^\alpha \theta^\nu, \quad h_{\alpha\bar{\beta}} = h_{\nu\bar{\mu}} u_\alpha{}^\nu u_{\bar{\beta}}{}^{\bar{\mu}}.$$

The pseudohermitian connection [6, 5].

Theorem ΨH

Given an admissible CR coframe (θ, θ^α), there are uniquely determined connection forms $\omega_\nu{}^\beta$, torsion forms $\tau^\alpha = A^\alpha{}_{\bar\mu}\theta^{\bar\mu}$ such that

$$\begin{aligned} d\theta &= ih_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta} \\ d\theta^\alpha &= \theta^\nu \wedge \omega_\nu{}^\alpha + \theta \wedge \tau^\alpha \\ dh_{\alpha\bar\beta} &= \omega_{\alpha\bar\beta} + \omega_{\bar\beta\alpha}. \end{aligned} \tag{12}$$

The connection forms satisfy

$$d\omega_\alpha{}^\beta = \omega_\alpha{}^\nu \wedge \omega_\nu{}^\beta + R_\alpha{}^\beta{}_{\nu\bar\mu}\theta^\nu \wedge \theta^{\bar\mu} + \dots . \tag{13}$$

(+ similar equation for the torsion forms.)

Remarks.

- $R_{\alpha\bar{\beta}\nu\bar{\mu}}$ is called the Tanaka-Webster curvature. Pseudohermitian (but not a CR) invariant.
- M is torsion free (i.e. $\tau^{\alpha} = 0$) $\iff$ the Reeb vector field is an infinitesimal CR automorphism.
- On a CR manifold M, there is a pseudohermitian structure that is torsion free $\iff$ there is a transverse infinitesimal CR automorphism; such M are called "rigid", "regular", or "Sasakian".
- The CR structure on M is in some sense the "conformal class" of pseudohermitian structures on M.

Tanaka-Webster curvature vs. CR curvature

Fix a pseudohermitian structure θ, and let (θ, θ^α) be an admissible coframe. Then $(\theta, \theta^\alpha, \theta^{\bar\beta}, \phi = 0)$ is a CTCM coframe. We pull down the CR curvature $S_{\alpha\bar\beta\nu\bar\mu}$ using this CTCM coframe.

Proposition

The CR curvature is the traceless part (Weyl tensor) of the Tanaka-Webster curvature; i.e.,

$$S_{\alpha\bar\beta\mu\bar\nu} = R_{\alpha\bar\beta\mu\bar\nu} - \frac{R_{\alpha\bar\beta}h_{\mu\bar\nu} + R_{\mu\bar\beta}h_{\alpha\bar\nu} + R_{\alpha\bar\nu}h_{\mu\bar\beta} + R_{\mu\bar\nu}h_{\alpha\bar\beta}}{n+2} + \frac{R(h_{\alpha\bar\beta}h_{\mu\bar\nu} + h_{\alpha\bar\nu}h_{\mu\bar\beta})}{(n+1)(n+2)},$$

where

$$R_{\alpha\bar\beta} := R_\mu{}^\mu{}_{\alpha\bar\beta} \text{ and } R := R_\mu{}^\mu$$

are respectively the *pseudohermitian Ricci* and *scalar curvatures* of (M, θ).

Further relations CR vs. Pseudohermitian.

Proposition

We have:

$$\phi_\beta{}^\alpha = \omega_\beta{}^\alpha + D_\beta{}^\alpha\theta, \quad \phi^\alpha = \tau^\alpha + D_\mu{}^\alpha\theta^\mu + E^\alpha\theta, \quad \psi = iE_\mu\theta^\mu - iE_{\bar\nu}\theta^{\bar\nu} + B\theta,$$

where

$$D_{\alpha\bar\beta} := \frac{iR_{\alpha\bar\beta}}{n+2} - \frac{iRg_{\alpha\bar\beta}}{2(n+1)(n+2)},$$

$$E^\alpha := \frac{2i}{2n+1}(A^{\alpha\mu}{}_{;\mu} - D^{\bar\nu\alpha}{}_{;\bar\nu}),$$

$$B := \frac{1}{n}(E^\mu{}_{;\mu} + E^{\bar\nu}{}_{;\bar\nu} - 2A^{\beta\mu}A_{\beta\mu} + 2D^{\bar\nu\alpha}D_{\bar\nu\alpha}).$$

C. Fefferman's approach. Just kidding!

The End

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