

# CR Geometry, Mappings into Spheres, and Sums-Of-Squares Lecture IV

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# Outline - Lecture VI

- 1 CR Mappings Between Spheres; The Poincaré-Alexander Theorem
- 2 From Faran to the HJY Gap Conjecture
- 3 A Sums-Of-Squares (SOS) Problem
- 4 SOS Conjecture  $\implies$  Gap Conjecture
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# The unit sphere $\mathbb{S}^n \subset \mathbb{C}^{n+1}$ .

- Let  $\mathbb{S}^n$  denote the unit sphere in  $\mathbb{C}^{n+1}$ .
- $\mathbb{S}^n$  is a compact strictly pseudoconvex CR manifold of CR dim  $n$ . (Superscript  $n$  refers to its CR dimension. The dimension is  $2n + 1$ .)

$$\sum_{j=1}^{n+1} |z_j|^2 - 1 = 0; \quad \|z\|^2 - 1 = 0.$$

- Notation: For  $a = (a_1, \dots, a_m)$ ,

$$\|a\|^2 := \sum_{j=1}^m |a_j|^2.$$

- $\mathbb{S}^n$  is locally the flat model for strictly pseudoconvex CR manifolds  $M^n$ .
- Understanding CR submanifold structure of  $\mathbb{S}^n$  is a fundamental problem. Start with locally spherical submanifolds.

# Automorphisms of $\mathbb{S}^N$ . Spherical equivalence of CR maps.

- $\text{Aut}(\mathbb{S}^N) \subset \text{Aut}(\mathbb{C}\mathbb{P}^N)$ ;  $T \in \text{Aut}(\mathbb{S}^N) \iff T$  rational map of degree one that preserves  $\mathbb{S}^N$ ; such  $T$  have no poles in the closed unit ball.
- $T \in \text{Aut}(\mathbb{S}^N)$  is called a "rigid motion" in  $\mathbb{S}^N$ .
- If  $\Sigma \subset \mathbb{S}^N$  is a submanifold, then we consider  $T(\Sigma) \subset \mathbb{S}^N$ , for  $T \in \text{Aut}(\mathbb{S}^N)$ , to be equivalent;  $T(\Sigma)$  and  $S(\Sigma)$  are **spherically equivalent** if  $S, T \in \text{Aut}(\mathbb{S}^N)$ .
- If  $f_1, f_2: M^n \rightarrow \mathbb{S}^N$  are CR maps, then  $f_1, f_2$  are **spherically equivalent** if  $f_2 = T \circ f_1 \circ \phi: M^n \rightarrow \mathbb{S}^N$  for  $T \in \text{Aut}(\mathbb{S}^N)$ ,  $\phi \in \text{Aut}(M^n)$ .  
**Notation:**  $f_1 \sim f_2$ .
- Important special case:  $M^n = \mathbb{S}^n$ ; or  $M^n \subset \mathbb{S}^n$ .

# Poincaré-Alexander Theorem.

## Poincaré-Alexander Theorem. ([17, 1])

Let  $M = M^n \subset \mathbb{S}^n$  be open, and  $f: M^n \rightarrow \mathbb{S}^n$  smooth, nonconstant CR map. Then  $f = T|_M$  for some  $T \in \text{Aut}(\mathbb{S}^n)$ .

## Theorem. (Forstneric, Cima-Suffridge[11, 3])

Let  $M = M^n \subset \mathbb{S}^n$  be open, and  $f: M^n \rightarrow \mathbb{S}^N$  smooth, nonconstant CR map. Then  $f = R|_M$  for some rational map  $R$  without poles on the closed unit ball.

## Theorem. (Faran, Cima-Suffridge [10, 2])

Let  $M = M^n \subset \mathbb{S}^n$  be open, and  $f: M^n \rightarrow \mathbb{S}^N$  smooth, nonconstant CR map. Assume  $N - n < n$ . Then  $f = (T \circ L)|_M$ , where  $T \in \text{Aut}(\mathbb{S}^N)$  and  $L: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{N+1}$  the standard linear embedding;  $L(z) = (z, 0)$ .

## CR maps $f: \mathbb{S}^n \rightarrow \mathbb{S}^N$ in higher codimension $N - n$ .

- $N - n < n$ . Rigidity:  $f = T \circ L$  (Webster, Faran, Huang, ...).
- $N - n = n$ . Two maps:  $f = T \circ L$  or  $f = T \circ W$  (Huang-Ji [14])

$$W(z_1, \dots, z_{n+1}) := (z_1, \dots, z_n, z_1 z_{n+1}, z_2 z_{n+1}, \dots, z_n^2 z_{n+1}).$$

- $N - n = n + 1$ . 1-parameter family (D'Angelo [4], Hamada [12]):

$$D_t(z_1, \dots, z_{n+1}) := (z_1, \dots, z_n, s_t z_{n+1}, c_t z_1 z_{n+1}, \dots, c_t z_n^2 z_{n+1}), \quad (1)$$

where  $c_t = \cos t$ ,  $s_t = \sin t$ ,  $t \in [0, \pi/2]$ .

- $N - n < 2n - 1$ . Nothing new!  $f = T \circ L \circ D_t$  (Huang-Ji-Xu [15]).
- $2n - 1 \leq N - n \leq 2n + 2$ . New maps: "generalized" Whitney maps + families.
- $N - n < 3n - 3$ . Nothing new!  $f = T \circ L \circ f_0$  (Huang-Ji-Yin [16]).

# Minimally embedded CR maps $f: M^n \rightarrow \mathbb{S}^N$ .

## Definition

- $f: M^n \rightarrow \mathbb{S}^N$  is **minimally embedded** if  $f(M^n)$  is not contained in a proper complex plane (affine subspace) section of  $\mathbb{S}^N$ , i.e.  $f$  does not "come from" a lower codimensional mapping  $f_0: M^n \rightarrow \mathbb{S}^{N_0}$  with  $N_0 < N$ ;  $f = T \circ L \circ f_0$ .
- If  $f(M^n)$  is contained in a proper complex plane section of dimension  $N_0 + 1$ , then we say that  $f$  is  **$N_0$ -flat**. I.e., " $f(M^n) \subset \mathbb{S}^{N_0}$ ".

# The HJY Gap Conjecture; $f: \mathbb{S}^n \rightarrow \mathbb{S}^N$ .

- $0 \leq N - n < n$ : All maps are  $n$ -flat (1st Gap; Faran '86, ...).
- $n + 1 < N - n < n + (n - 1) = 2n - 1$ : All maps are  $2n + 1$ -flat (2nd Gap; Huang-Ji-X0 '06).
- $2n + 2 < N - n < n + (n - 1) + (n - 2) = 3n - 3$ : All maps are  $3n + 2$ -flat (3rd Gap; Huang-Ji-Yin '12).
- $\vdots$
- $(k - 1)n + k - 1 < N - n < n + (n - 1) + \dots + (n - k + 1)$ : All maps are  $kn + k - 1$ -flat ( $k$ th Gap; conjectural for  $k \geq 4$ ).
- $\vdots$
- After that ( $k = k_0 \sim \sqrt{2n}$ ): No more gaps (eventually D'Angelo-Han-Lebl '07 [5]).



# A Sums-Of-Squares (SOS) Problem.

- Let  $A(z, \bar{z})$  be a Hermitian (real) polynomial in  $(z, \bar{z})$ , with  $z \in \mathbb{C}^n$ .
- Linear algebra  $\implies A(z, \bar{z}) = \|p(z)\|^2 - \|q(z)\|^2$ , where  $p = (p_1, \dots, p_s)$ ,  $q = (q_1, \dots, q_t)$  are linearly independent polynomials in  $z$ .
- The **rank** of  $A$  is  $\text{rank } A = s + t$ .
- $A$  is an **SOS** if  $A(z, \bar{z}) = \|p(z)\|^2$ .

**An SOS Problem.** Assume  $A(z, \bar{z})\|z\|^2$  is an SOS. What are possible ranks of  $A(z, \bar{z})\|z\|^2$ ?

Huang's Lemma ([13])

If  $A(z, \bar{z})\|z\|^2$  is an SOS, then  $\text{rank } A\|z\|^2 = 0$  or  $\geq n$ .

# The SOS Conjecture.

The Gap Conjecture follows from the following SOS Conjecture, in view of a result of the speaker ([7]).

## SOS Conjecture

For any Hermitian polynomial  $A(z, \bar{z})$  such that  $\|z\|^2 A(z, \bar{z})$  is an SOS, the rank  $r$  of  $\|z\|^2 A(z, \bar{z})$  can only take the values:

$$\begin{aligned} &0, \\ &n, \\ &2n - 1, 2n, \\ &\vdots \end{aligned} \tag{2}$$

$$n + (n - 1) + \dots + (n - k + 1), \dots, kn,$$

or

$$\geq n + (n - 1) + \dots + (n - k_0 + 1) = k_0 n - k_0(k_0 - 1)/2$$

## Second Fundamental Form and Covariant Derivatives

For  $M = M^n$  and  $\widehat{M} = \widehat{M}^N$  strictly  $\psi$ cvx,  $f: M \rightarrow \widehat{M}$  transversal CR immersion,  $p \in M$ ,

$$\text{SFF}_p: T_p^{1,0}M \times T_p^{1,0}M \rightarrow T_{f(p)}^{1,0}\widehat{M}/f_*(T_p^{1,0}M),$$

and, for  $m \in \mathbb{N}$ ,

$$\nabla^m \text{SFF}_p: T_p^{1,0}M \times \dots \times T_p^{1,0}M \rightarrow T_{f(p)}^{1,0}\widehat{M}/f_*(T_p^{1,0}M).$$

Given local frames  $L_1, \dots, L_n$ , and  $\widehat{L}_1, \dots, \widehat{L}_{N-n}$  for  $T^{1,0}M \cong \mathbb{C}^n$  and  $T^{1,0}\widehat{M}/f_*(T^{1,0}M) \cong \mathbb{C}^{N-n}$ ,

$$\text{SFF}_p \sim \omega_\alpha^a{}_\beta, \quad \nabla^m \text{SFF}_p \sim \omega_\alpha^a{}_{\beta;\gamma_1 \dots \gamma_m},$$

where  $\alpha, \beta, \gamma_j \in \{1, \dots, n\}$  and  $a \in \{1, \dots, N-n\}$ .

# Span of SFF; Spaces $E_m(p)$

Introduce:

$$E_2(p) \subset E_3(p) \subset \dots \subset E_m(p) \subset \dots \subset \mathbb{C}^{N-n},$$

by

$$E_m(p) := \text{span}_{\mathbb{C}} \left\langle \left( \omega_{\alpha_1}^a \alpha_2; \alpha_3 \dots \alpha_j \right)_{a=1}^{N-n} : \alpha_1, \dots, \alpha_j \in \{1, \dots, n\}, j \leq m \right\rangle.$$

## Definition

- $f: M^n \rightarrow \widehat{M}^N$  is  **$(m, d)$ -nondegenerate** at  $p \in M$  if  $\dim E_m(p) = d$ , where  $E_{m-1}(p) \subsetneq E_m(p) = E_{m+1}(p) = \dots$
- Set  $d_l(p) := \dim E_l(p)$ ,  $l = 2, 3, \dots, m$ ; for convenience, also set  $d_1(p) = 0$ . Note:  $d = d_m(p)$ .

## Note:

- $0 = d_1(p) \leq d_2(p) < \dots < d_{m-1}(p) < d_m(p) = d(p) \leq N - n$ .
- At generic  $p$ ,  $d_l(p)$  are locally constant  $= d_l$ .

# Flat Maps; $f: M^n \rightarrow S^N \subset \mathbb{C}^{N+1}$

## Observation 1:

If  $f(M^n)$  is contained in a complex plane  $P = P^{N_0+1} \subset \mathbb{C}^{N+1}$ , then  $d(p) \leq N_0 - n$  for all  $p \in M$ .

Converse? Well, if  $d(p)$  is "small":

## Theorem ([8, 9])

If  $d(p) \leq d$  for  $p \in M$  and  $d < n$ , then  $f(M^n)$  is contained in a complex plane  $P = P^{N_0+1}$  with  $N_0 - n = d$  (i.e.  $N_0 = n + d$ ).

And if  $d(p)$  is larger?

## "Observation" 2

Let  $f = D_t: S^n \rightarrow S^N$ , with  $D_t$  given by (1) and  $N - n = n + 1$ . Then,  $d(p) \leq n$  for  $p \in S^n$ , but  $f(S^n)$  is **minimally embedded** (not contained in a plane  $P = P^{2n+1} \subset \mathbb{C}^{2n+2}$ ), except for  $t = 0$  (Whitney) and  $t = \pi/2$  (linear).

# The $d_l$ measure flatness.

## Theorem 1 (E. '13 [7])

Let  $f: M^n \rightarrow S^N$ , and assume that there are integers

$$0 \leq k_2, k_3, \dots, k_m \leq n - 1$$

( $m = \text{degeneracy of } f$ ) such that, for  $p \in M$ :

$$d_l(p) - d_{l-1}(p) < \sum_{i=0}^{k_l} (n - i), \quad l = 2, \dots, m, \quad (d_1(p) = 0).$$

If  $k < n$ , where

$$k := \sum_{l=2}^m k_l, \quad d := \max_{p \in M} d(p)$$

then  $f(M)$  is contained in a complex plane  $P^{N_0+1}$  with  $N_0 - n = d + k$ .

## Total second fundamental polynomial of $f: M^n \rightarrow \mathbb{S}^N$ .

- For  $l \geq 2$ , set  $\Omega_{(l)} = (\Omega_{(l)}^1, \dots, \Omega_{(l)}^{N-n})$ , where

$$\Omega_{(l)}^j(z) := \omega_{\gamma_1 \gamma_2 \dots \gamma_l}^a z^{\gamma_1} \dots z^{\gamma_l}, \quad a = n + j, \quad (3)$$

- The **total second fundamental polynomial**  $\Omega = (\Omega^1, \dots, \Omega^{N-n})$  of  $f$  is

$$\Omega^j(z) := \sum_{l=2}^m \Omega_{(l)}^j(z), \quad (4)$$

where  $m$  is the integer where the dimensions  $d_l$  stabilize.

- Linear algebra  $\implies d = \text{rank } \|\Omega(z)\|^2$ .

### Total Polynomial Gauss Equation for $f: \mathbb{S}^n \rightarrow \mathbb{S}^N$ ([6])

There exists a Hermitian polynomial  $A(z, \bar{z})$  such that

$$\|\Omega(z)\|^2 = A(z, \bar{z}) \|z\|^2. \quad (5)$$



# The SOS Conjecture $\implies$ the Gap Conjecture.

**Sketch of proof ([6]).** Let  $f: \mathbb{S}^n \rightarrow \mathbb{S}^N$  and assume that  $N - n$  is in  $p$ th gap

$$(p-1)n + p - 1 < N - n < n + (n-1) + \dots + (n-p+1), \quad p \leq k_0.$$

Recall: Gap Conjecture predicts that  $f$  is  $(pn + p - 1)$ -flat. The rank of nondegeneracy  $d \leq N - n < n + (n-1) + \dots + (n-p+1)$ . Total Polynomial Gauss Equation and SOS Conjecture  $\implies d \leq (p-1)n$ .

## Proposition

There exist  $k_2, \dots, k_m$ ,  $k = k_2 + \dots + k_m$  as in Theorem 1 such that  $k \leq p - 1$ .

Theorem 1  $\implies f$  is  $(n + d + k)$ -flat; and

$$n + d + k \leq n + (p-1)n + (p-1) = pn + p - 1.$$



Thank You for Your Attention!



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