

CR Geometry, Mappings into Spheres, and Sums-Of-Squares Lecture I

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- **Lecture I.** An introduction to CR geometry. Non-existence of a Riemann mapping theorem in \mathbb{C}^n for $n \geq 2$; a motivational example. Basic notions and invariants in CR geometry. The Levi form.
- **Lecture II.** The flat models. Levi nondegenerate hypersurfaces. Normal forms, principal G-bundles and Cartan connections, pseudohermitian geometry.
- **Lecture III.** CR immersions into flat models. The CR second fundamental form and the Gauss equation. Analysis of the Gauss equation. Basic rigidity theorems.
- **Lecture IV.** Mappings of spheres into spheres. The HJY Gap Conjecture. Sums-Of-Squares of polynomials.

Outline - Lecture I

- 1 No Riemann Mapping Theorem in higher dimensions
- 2 A Riemann Mapping Theorem in higher dimensions
- 3 CR structure of the boundary of a complex manifold
- 4 The Levi form.
- 5 References

Biholomorphic equivalence of domains in \mathbb{C}^m .

Classification of domains in the complex plane \mathbb{C} rests on the following corner stone in complex analysis:

Riemann Mapping Theorem

Let $\Omega \subset \mathbb{C}$ be a simply connected domain with $\Omega \neq \mathbb{C}$. Then, there exists a biholomorphism $f: \Omega \rightarrow \mathbb{D}$, where $\mathbb{D} := \{z \in \mathbb{C}: |z| < 1\}$ denotes the unit disk.

This is no longer true in higher dimensions.

Theorem 0 (Poincaré)

There is no biholomorphic mapping $f: \mathbb{D}^2 = \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{B}_2$, where $\mathbb{D}^2 := \{(z, w): |z| < 1, |w| < 1\}$ is the unit bidisk in \mathbb{C}^2 and $\mathbb{B}_2 := \{(z, w): |z|^2 + |w|^2 < 1\}$ the unit ball.

Poincaré's proof of Theorem 0.

Suppose there exists a biholomorphism $f: \mathbb{D}^2 \rightarrow \mathbb{B}_2$.

$$\implies \exists \text{ isomorphism } f^c: \text{Aut}(\mathbb{B}_2) \rightarrow \text{Aut}(\mathbb{D}^2), \quad f^c\phi = f^{-1} \circ \phi \circ f.$$

Poincaré computed $\text{Aut}(\mathbb{B}_2)$, and $\text{Aut}(\mathbb{D}^2)$:

$$\text{Aut}(\mathbb{B}_2) \cong SU(2, 1)/\sim, \quad \text{Aut}(\mathbb{D}^2) = (SU(1, 1)/\sim)^2.$$

In particular,

$$\dim_{\mathbb{R}} \text{Aut}(\mathbb{B}_2) = 8, \quad \dim_{\mathbb{R}} \text{Aut}(\mathbb{D}^2) = 6,$$

which means they cannot be isomorphic. □

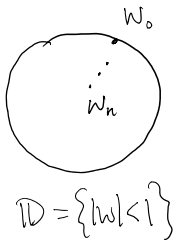
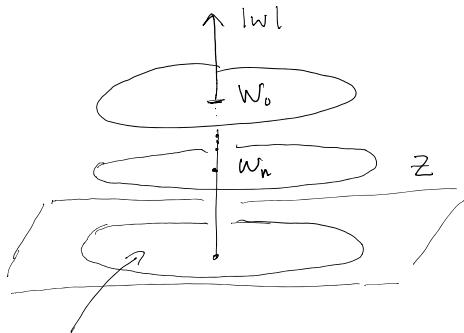
"CR" approach to Theorem 0.

Assume that there exists a biholomorphic (or just proper holomorphic) mapping $f: \mathbb{D}^2 \rightarrow \mathbb{B}_2$.

- Show that the holomorphic mapping induces a "partially holomorphic mapping" (CR) of the boundaries $f_0: \partial(\mathbb{D}^2) \rightarrow \partial\mathbb{B}_2$.
- Show that the boundaries have different "invariants" preserved by f_0 ; in this case, $\partial(\mathbb{D}^2)$ contains non-trivial complex curves, but $\partial\mathbb{B}_2$ does not.
- Conclude that no such mapping f can exist.

Bidisk $\mathbb{D}^2 \subseteq \mathbb{C}^2$

Sunday, August 30, 2015 6:32 AM



$$\mathbb{D} = \{|z| < 1\}$$

$$\partial(\mathbb{D}^2) = \{|z| < 1, |w| = 1\} \cup \{|z| = 1, |w| < 1\} \cup \{|z| = 1, |w| = 1\}$$

"CR" Proof of Theorem 0.

Suppose there exists a proper holomorphic mapping $f: \mathbb{D}^2 \rightarrow \mathbb{B}_2$;

$$f(z, w) = (f^1(z, w), f^2(z, w)).$$

Pick $w_0 \in \partial\mathbb{D}$, i.e., $|w_0| = 1$, and $w_n \in \mathbb{D}$ with $w_n \rightarrow w_0$. Set

$$A_n(z) = (A_n^1(z), A_n^2(z)) := (f^1(z, w_n), f^2(z, w_n)).$$

We note that $A_n^i(z)$ are holomorphic in \mathbb{D} , and $|A_n^i| \leq 1$. By Montel's Theorem, we may assume (by going to a subsequence) that there are holomorphic functions $A_0^i(z)$ in \mathbb{D} such that $A_n^i \rightarrow A_0^i$.

Claim. $\|A_0(z)\|^2 := |A_0^1(z)|^2 + |A_0^2(z)|^2 = 1$.

Proof. By properness! $(z, w_n) \rightarrow \partial(\mathbb{D}^2) \implies \|A_n(z)\|^2 \rightarrow 1$. □

$\implies A_0: \mathbb{D} \rightarrow \partial\mathbb{B}_2$ is a holomorphic map (analytic disk).

"CR" Proof of Theorem 0; continued.

Lemma (No analytic disks in $\partial\mathbb{B}_2$)

If $A_0: \mathbb{D} \rightarrow \partial\mathbb{B}_2$ is holomorphic, then $A_0(z)$ is constant.

Proof. Use $\text{Aut}(\mathbb{B}_2)$! By replacing A_0 with UA_0 , $U \in SU(2)$, we may assume that $A_0(0) = (1, 0)$.

- $\|A_0(z)\|^2 = 1 \implies |A_0^1(z)|$ has maximum at $z = 0$.
- Maximum Principle $\implies A_0^1(z)$ is constant, so $A_0^1(z) = 1$.
- $\|A_0(z)\|^2 = 1 \implies |A_0^2(z)|$ is identically 0.

□

We shall obtain a contradiction (proving Theorem 0) by showing

$$\frac{\partial f}{\partial z} = 0 \implies f \text{ not proper.}$$

End of proof of Theorem 0; $\partial f / \partial z = 0$.

Fix $z = z_0 \in \mathbb{D}$. Note that, for $j = 1, 2$,

$$\frac{\partial f^j}{\partial z}(z_0, w) = \frac{1}{2\pi i} \int_{|\zeta|=r < 1} \frac{f^j(\zeta, w) d\zeta}{(\zeta - z_0)^2}$$

is bounded as a function of $w \in \mathbb{D}$. Thus, there are nontangential limits $h = (h^1, h^2)$, with $h^j \in L^\infty(\partial\mathbb{D})$, such that for a.e. $w_0 \in \partial\mathbb{D}$,

$$h(w_0) = \lim_{w \rightarrow w_0} \frac{\partial f}{\partial z}(z_0, w).$$

For $w_n \rightarrow w_0$ as before, we have $\partial f / \partial z(z_0, w_n) = A'_n(z_0) \rightarrow A'_0(z_0) = 0$ since $A_0(z)$ is constant. It follows that the nontangential limit at w_0 vanishes: $h(w_0) = 0$. Since this holds for all w_0 where the nontangential limits exist (a.e.), a standard uniqueness result implies

$$\frac{\partial f}{\partial z}(z_0, w) = 0.$$



Local biholomorphic equivalence of submanifolds.

Equivalence fails locally!

Proposition

Let $U \subset \mathbb{C}^2$ be an open neighborhood of $(z_0, w_0) \in \mathbb{C}^2$ with $|z_0| < 1$ and $|w_0| = 1$. If $f: U \rightarrow \mathbb{C}^2$ is a holomorphic mapping such that $f(\partial\mathbb{D}^2 \cap U) \subset \partial\mathbb{B}_2$, then $f = f(w)$.

Definition

Let $M_1, M_2 \subset \mathbb{C}^m$ be real submanifolds with $p_1 \in M_1, p_2 \in M_2$. If there exist an open neighborhood $U \subset \mathbb{C}^n$ of p_1 and a biholomorphic mapping $f: U \rightarrow f(U) \subset \mathbb{C}^n$ such that $f(p_1) = p_2$ and $f(M_1 \cap U) = M_2 \cap f(U)$, then (M_1, p_1) and (M_2, p_2) are said to be **biholomorphically equivalent** (BHE).

$$(M_1, p_1) \cong_{BHE} (M_2, p_2).$$

Remark: Different notions of equivalence! Analytic vs. smooth vs. formal.

Remarks on BHE.

- All real-analytic curves in \mathbb{C} are locally BHE. $(\gamma, p) \cong_{BHE} (\mathbb{R}, 0)$.
- Real hypersurfaces in \mathbb{C}^m , $m \geq 2$, are in general not locally BHE. $(\partial\mathbb{D}^2, p_1) \not\cong_{BHE} (\partial\mathbb{B}_2, p_2)$.
- For $m \geq 2$, a real hypersurface $M \subset \mathbb{C}^m$ is in general not BHE to *itself* at two different points. $(M, p_1) \not\cong_{BHE} (M, p_2)$ if $p_1 \neq p_2$.
- But $\partial\mathbb{B}_m$ is. $(\partial\mathbb{B}_m, p_1) \cong_{BHE} (\partial\mathbb{B}_m, p_2)$, for all $p_1, p_2 \in \partial\mathbb{B}_m$. Such manifolds are called **homogeneous**.

Definition

A real-analytic hypersurface $M \subset \mathbb{C}^m$ is **locally spherical** at $p_1 \in M$ if $(M, p_1) \cong_{BHE} (\partial\mathbb{B}_m, p_2)$. (For smooth M use smooth CR equivalence.)

A Riemann Mapping Theorem in higher dimensions.

Theorem (S.-S. Chern – S. Ji, '96 [2])

Let $\Omega \subset \mathbb{C}^m$ be a bounded, simply connected domain. If $\partial\Omega$ is locally spherical, then there exists a biholomorphic mapping $f: \Omega \rightarrow \mathbb{B}_m$.

Remarks:

- If $\partial\Omega$ is not real-analytic, but smooth, then "locally spherical" at $p \in \partial\Omega$ can be defined as the existence of $U \subset \mathbb{C}^n$ and a smooth mapping $f: U \cap \bar{\Omega} \rightarrow \mathbb{C}^n$ such that $f: U \cap \Omega \rightarrow f(U \cap \Omega)$ is biholomorphic and $f(\partial\Omega \cap U) \subset \partial\mathbb{B}_n$.
- In the real-analytic case, it in fact suffices that $\partial\Omega$ is locally spherical at some point $p \in \partial\Omega$. The local biholomorphism then extends as a global biholomorphism $f: \Omega \rightarrow \mathbb{B}_m$.
- X. Huang and S. Ji [4] have proved a Riemann Mapping Theorem for a more general class of domains, where again the assumption is that the boundaries are locally equivalent.

Recall [1]: Complex structure on a vector space.

A **complex structure** on \mathbb{R}^{2m} is a linear map $J = J_p: T_p\mathbb{R}^{2m} \rightarrow T_p\mathbb{R}^{2m}$ such that $J^2 = -I$. J extends by linearity to $\mathbb{C}T_p\mathbb{R}^{2m} := \mathbb{C} \otimes T_p\mathbb{R}^{2m}$ and splits it into an $i = \sqrt{-1}$ and $-i$ eigenspace,

$$\mathbb{C}T_p\mathbb{R}^{2m} = T_p^{1,0}\mathbb{R}^{2m} \oplus T_p^{0,1}\mathbb{R}^{2m}$$

with $T_p^{0,1}\mathbb{R}^{2m} = \overline{T_p^{1,0}\mathbb{R}^{2m}}$. The standard complex structure in coordinates $(x_1, y_1, \dots, x_m, y_m)$ is given by

$$J(\partial/\partial x_j) = \partial/\partial y_j, \quad J(\partial/\partial y_j) = -\partial/\partial x_j,$$

and $T_p^{1,0}\mathbb{R}^{2m}$ is spanned by $\partial/\partial z_1, \dots, \partial/\partial z_m$,

$$\frac{\partial}{\partial z_j} := \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right).$$

The standard linear structure yields \mathbb{C}^m with complex coordinates

$$z = (z_1, \dots, z_m), \quad z_j = x_j + iy_j.$$

CR structure of a real hypersurface in a complex manifold.

Let $\Omega \subset \mathbb{C}^m$ be a domain with complex coordinate $z = (z_1, \dots, z_m)$. Let $M \subset \Omega$ be a real hypersurface; i.e., defined locally near every $p \in M$ by

$$M \cap V_p := \{z \in V_p : \rho(z, \bar{z}) = 0\},$$

where $p \in V_p \subset \Omega$, $\rho \in C^\kappa(V_p, \mathbb{R})$, $d\rho|_M \neq 0$. For us, κ is either ∞ ("smooth") or ω ("real-analytic").

Definition. The **CR tangent space** to M at $p \in M$ is given by

$$T_p^{0,1}M := \mathbb{C}T_pM \cap T_p^{0,1}\Omega; \quad T_p^{1,0}M := \overline{T_p^{0,1}M}.$$

$$L = \sum_{j=1}^m a_j \frac{\partial}{\partial z_j} \in T_p^{1,0}M \iff \sum_{j=1}^m \frac{\partial \rho}{\partial z_j}(p, \bar{p}) a_j = 0.$$

CR manifolds of hypersurface type.

- $T_p^{0,1}M$ is a complex hyperplane in the m -dimensional complex vector space $T_p^{0,1}\Omega$. Thus, $\dim_{\mathbb{C}} T_p^{0,1}M = m - 1$ for all $p \in M$.
- Set $n = m - 1$; $M \subset \Omega \subset \mathbb{C}^{n+1}$, $\dim_{\mathbb{R}} M = 2(n + 1) - 1 = 2n + 1$.
- $T_p^{0,1}M$ form a rank n sub-bundle $T^{0,1}M$ of the complexified tangent bundle $\mathbb{C}TM$ (of rank $2n + 1$). Sections of $T^{0,1}M$ are called **CR vector fields**.

The following properties of $T^{0,1}M$ are fundamental:

(P1) $T_p^{1,0}M \cap T_p^{0,1}M = \{0\}$;

(P2) $[T^{0,1}M, T^{0,1}M] \subset T^{0,1}M$; i.e., if X, Y are CR vector fields, then the commutator $[X, Y]$ is a CR vector field.

Note: (P1) $\implies T_M^{1,0} \oplus T_p^{0,1}M$ is a complex hyperplane in $\mathbb{C}T_pM$. $\implies \mathbb{C}T_pM = T_p^{1,0}M \oplus T_p^{0,1}M \oplus \mathbb{C}\langle X_p \rangle$, if $X_p \in \mathbb{C}T_pM \setminus T_M^{1,0} \oplus T_p^{0,1}M$.

Definition. M is a **CR manifold** (of hypersurface type) with CR bundle $T^{0,1}M$; $\text{CR dim } M := \dim_{\mathbb{C}} T_p^{0,1}M = n$.

The first invariant of a CR manifold; the Levi form.

- Since $T^{1,0}M \oplus T^{0,1}M$ is a Hermitian sub-bundle of corank 1 in $\mathbb{C}TM$, it can be defined by a real 1-form θ . Such θ is called a "contact form". Any other is of form $\tilde{\theta} = a\theta$, $a \neq 0$ real function.
- On Ω , the differential $d = \partial + \bar{\partial}$, where

$$\partial u := \sum_{j=1}^{n+1} \frac{\partial u}{\partial z_j} dz_j.$$

- By definition, $T_p^{1,0}M = \{X_p \in T_p^{1,0}\mathbb{C}^{n+1} : \langle \partial\rho, X_p \rangle = 0\}$. Since $0 = d\rho = \partial\rho + \bar{\partial}\rho$ on M ,

$$\theta := i\partial\rho|_M = -i\bar{\partial}\rho|_M = \overline{i\partial\rho|_M}$$

is real and annihilates $T^{1,0}M \oplus T^{0,1}M$. Thus, $\theta = i\partial\rho|_M$ is a contact form on M .

The Levi form.

Definition: The Levi form $\mathcal{L} = \mathcal{L}_p^\theta$ of M at p is the Hermitian form

$$\mathcal{L}(X_p, Y_p) := \frac{i}{2} \langle \theta, [X, \bar{Y}] \rangle|_p, \quad X_p, Y_p \in T_p^{1,0}M.$$

where X, Y are local sections of $T^{1,0}M$ (anti-CR vector fields) extending X_p, Y_p .

- Independent of extensions (well-defined) by Cartan's identity:

$$\langle \omega, [Z, V] \rangle = -2 \langle d\omega, Z \wedge V \rangle + Z \langle \omega, V \rangle - V \langle \omega, Z \rangle.$$

$$\implies \mathcal{L}(X_p, Y_p) = -i \langle d\theta, X_p \wedge \bar{Y}_p \rangle.$$

- If $\tilde{\theta} = a\theta$, then $\mathcal{L}_p^{\tilde{\theta}} = a(p)\mathcal{L}_p^\theta$ by

$$d\tilde{\theta} = ad\theta + da \wedge \theta.$$

In local coordinates: $M \subset \Omega \subset \mathbb{C}^{n+1}$.

Choose local coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ in Ω , vanishing at $p \in M$:

$$\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w), \quad \phi(z, 0, s) = \phi(0, \zeta, s) = O\left(\|(z, s)\|^{2K}\right).$$

Note: $d\phi(0) = 0 \implies \mathbb{C}T_0M = \mathbb{C}\langle \partial/\partial z_j|_0, \partial/\partial \bar{z}_j|_0, \operatorname{Re}(\partial/\partial w|_0) \rangle$.

As a local frame for $T^{1,0}M$:

$$L_j := \frac{\partial}{\partial z_j} + \frac{2i\phi_{z_j}}{1 - i\phi_s} \frac{\partial}{\partial w}, \quad j = 1, \dots, n.$$

With $\rho = \operatorname{Im} w - \phi$, we may choose

$$\theta = i\partial\rho|_M = \frac{1}{2}(1 - i\phi_s)dw|_M - i \sum_{j=1}^n \phi_{z_j} dz_j|_M.$$

We may choose $(z, s) \in \mathbb{C}^n \times \mathbb{R}$ as local chart on M :

$$(z, w) \mapsto (z, w = s + i\phi(z, \bar{z}, s)).$$

CR functions and mappings.

Definition: A function h on M is **CR** if $\bar{L}h = 0$ for all CR vector fields \bar{L} .

As a mapping $h: M \rightarrow \mathbb{C}$,

$$h \text{ is CR} \iff h_*(T^{0,1}M) \subset T^{0,1}\mathbb{C}.$$

$$\zeta = h(z, \bar{z}, s) \implies h_*(\bar{L}) = (\bar{L}h) \frac{\partial}{\partial \zeta} + (\bar{L}\bar{h}) \frac{\partial}{\partial \bar{\zeta}}.$$

Definition: A mapping $f: M \rightarrow M'$ is **CR** if $f_*(T^{0,1}M) \subset T^{0,1}M'$.

If $M' \subset \Omega' \subset \mathbb{C}^{m'}$, then:

$$f = (f_1, \dots, f_{m'}) : M \rightarrow M' \text{ is CR} \iff \text{each } f_j \text{ is a CR function.}$$

Basic Example: The restriction (or boundary value) of a holomorphic function/mapping to M is CR. The converse is an interesting problem, but will not be addressed in these lectures.

Invariance of Levi form under CR mapping $f : M \rightarrow M'$.

Pick contact forms θ, θ' on M, M' . Definition of CR $\implies f^*\theta' = a\theta$. For $X_p, Y_p \in T_p^{1,0}M$,

$$\begin{aligned}(\mathcal{L}')_{f(p)}^{\theta'}(f_*X_p, f_*Y_p) &= \frac{i}{2} \langle \theta', [f_*X, \overline{f_*Y_p}] \rangle = -i \langle d\theta', f_*X_p \wedge \overline{f_*Y_p} \rangle \\ &= -i \langle f^*d\theta', X_p \wedge \overline{Y_p} \rangle = -i \langle d(a\theta), X_p \wedge \overline{Y_p} \rangle \\ &= \mathcal{L}_p^{a\theta}(X_p, Y_p) = a(p) \mathcal{L}_p^\theta(X_p, Y_p).\end{aligned}$$

In a local frame L_1, \dots, L_n and contact form θ , $T_p^{1,0}M \cong \mathbb{C}^n$,

$$\mathcal{L}^\theta(x, y) = xEy^*, \quad x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{C}^n,$$

where $E = E_p^\theta$ is the Hermitian $n \times n$ matrix with matrix elements $E_{jk} = i/2 \langle \theta, [L_j, \bar{L}_k] \rangle$. If $f_* \cong B$, $n \times n'$ matrix, then *Levi form Invariance*:

$$aE = BE'B^*.$$

Levi nondegenerate CR manifolds.

- Definitions.** 1) M is **Levi nondegenerate** at p if the Levi form \mathcal{L}_p is nondegenerate: $Y_p \mapsto \mathcal{L}_p(\cdot, Y_p) \in (T^{1,0}M)^*$ is injective; $\iff \det E_p \neq 0$.
- 2) M is **strictly pseudoconvex** at p if \mathcal{L}_p is positive definite (for some θ).

The "opposite" of Levi nondegenerate is Levi flat. M is **Levi flat** if the Levi form $\mathcal{L}_p = 0$ for all $p \in M$.

Proposition

M is Levi flat $\iff M$ is foliated by complex manifolds Σ_t with $\dim \Sigma_t = CR \dim M = n$.

Proof. $\mathcal{L}_p = 0 \implies \operatorname{Re}(T^{1,0}M \oplus T^{0,1}M)$ involutive. Frobenius Theorem $\implies M$ foliated by Σ_t with $T\Sigma_t = \operatorname{Re}(T^{1,0}M \oplus T^{0,1}M)$.

Newlander-Nirenberg Theorem $\implies \Sigma_t$ are complex manifolds. Converse is easy. See [1] for FT and NNT. □

Remark. $\partial\mathbb{B}_2$ is strictly pseudoconvex, and $\partial\mathbb{D}^2$ is Levi flat (at smooth points).

Levi form in terms of a defining function ρ .

- $T_p^{1,0}M \subset T_p^{1,0}\mathbb{C}^{n+1} \cong \mathbb{C}^{n+1}$ and $x \in T_p^{1,0}M$ if

$$\sum_{j=1}^{n+1} \frac{\partial \rho}{\partial z_j}(p) x_j = 0.$$

- Choose $\theta = i\partial\rho|_M$.

$$\begin{aligned} \mathcal{L}^\theta(X, Y) &= i\langle d\theta, X \wedge Y \rangle = \langle \partial\bar{\partial}\rho, X \wedge Y \rangle \\ &= \sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) x_j \bar{x}_k. \end{aligned}$$

- Thus, $\mathcal{L}^\theta \sim (n+1) \times (n+1)$ matrix $F = (\partial^2 \rho / \partial z_k \partial \bar{z}_k)$, restricted to n -dimensional subspace $T_p^{1,0}M \subset \mathbb{C}^{n+1}$.

Fefferman's complex Monge-Ampère Operator.

Consider the complex Monge-Ampère type operator in \mathbb{C}^{n+1} :

$$J(u) := (-1)^{n+1} \det \begin{pmatrix} u & u_{\bar{z}} \\ u_z & u_{z\bar{z}} \end{pmatrix}.$$

Proposition

Let $M \subset \Omega \subset \mathbb{C}^{n+1}$ be defined by $\rho = 0$. Then, M is Levi nondegenerate at $p \in M \iff J(\rho)|_p \neq 0$.

Proof. Let $p \in M$, and F denote $(n+2) \times (n+2)$ matrix such that $J(\rho) = (-1)^{n+1} \det F$. Pick $\tilde{x} = (c, x) \in \mathbb{C} \times \mathbb{C}^{n+1}$. Then, $\tilde{x}F = 0 \iff x\rho_z = 0$ and $c\rho_{\bar{z}} + x\rho_{z\bar{z}} = 0$. Note that there is $c \in \mathbb{C}$ such that $c\rho_{\bar{z}} + x\rho_{z\bar{z}} = 0 \iff x\rho_{z\bar{z}}y^* = 0$ for all $\rho_{\bar{z}}y^* = 0$. \square

Fefferman's defining equations.

Theorem (Fefferman, '76 [3])

Let $M \subset \Omega \subset \mathbb{C}^{n+1}$ be strictly pseudoconvex, defined by $\rho = 0$. Then, there is a unique, mod $O(\rho^{n+3})$, defining function r for M such that $J(r) = 1 + O(\rho^{n+2})$.

Remark: Such r is called a **Fefferman defining function** for M . Useful for studying invariants of strictly pseudoconvex domains, notably their Bergman kernels.



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