Def. A domain $\Omega \subseteq \mathbb{C}^n$ is a domain of holomorphy if $\exists \Omega_1, \Omega_2 \subseteq \mathbb{C}^n$ s.t. $\Omega_1 \cap \Omega_2$ and 
$\forall f \in \mathcal{O}(\Omega_1) \exists f_2 \in \mathcal{O}(\Omega_2)$ s.t. $f = f_2$ in $\Omega_1$.

Loosely, for all $\Omega_2$ s.t. $\Omega \cap \Omega_2 \neq \emptyset$, $\exists f \in \mathcal{O}(\Omega)$ that does not extend into $\Omega_2$.

Def. Let $K \subseteq \Omega$, $\Lambda(\Omega) =$ class of function on $\Omega$; $\Lambda(\Omega)$-hull of $K$ is $\{ z \in \Omega : |f(z)| < \sup_{K} f \}$

- $\mathcal{O}(\Omega)$-hull is holomorphic hull, $\mathcal{O}_\Omega$
- $\Omega = \mathbb{C}^n$, $\mathcal{O}[\mathbb{C}^n]$-hull is polynomial hull, $\mathcal{P}_\Omega$

\[ \text{Ob:} \]

1. $\Lambda_1(\Omega) = \Lambda_2(\Omega) \Rightarrow \Lambda_2(\Omega)$-hull $\subseteq \Lambda_1(\Omega)$-hull
2. $\Omega = \mathbb{C}^n; \Lambda(\mathbb{C}^n) = \{ f_3(z) = e^{2iz}; z \in \mathbb{C}^n \} \Rightarrow \Lambda(\mathbb{C}^n)$-hull
= convex hull.

3. \( \mathbb{D} + \mathcal{O} \Rightarrow \) For any \( K \subset \Omega \), \( \mathbb{D} \subset \text{convex hull of } K \).

4. \( \Lambda(\Omega) = \mathcal{P}(\Omega) \Rightarrow \Lambda(\Omega) \)-hull is closed (in \( \Omega \)).

**Thm 1.** The following are equivalent:

(i) \( \Omega \subset \mathcal{O}^n \) is domain of holomorphy

(ii) \( K \subset \Omega \Rightarrow \mathcal{P} \subset \text{cc} \Omega \).

(iii) \( \exists f \in \mathcal{O}(\Omega) \) that does not extend across \( \partial \Omega \); i.e. \( \exists \Omega_1 \subset \Omega_2, f_2 \in \mathcal{O}(\Omega_2) \), w/ \( f = f_2 \) in \( \Omega_1 \).

**Proof:** (iii) \( \Rightarrow \) (i) obviously. Prove (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii).

(i) \( \Rightarrow \) (ii), let \( D_r \) be the polydisk \( D_r = \{ z : |z_i| < r \} \), and

\[
\Delta_\Omega(z) = \sup \{ r > 0 : \{|z|^2 + D_r \subset \Omega} \}.
\]

\( K \subset \Omega \Rightarrow \inf_{K} \Delta_\Omega = r_0 > 0 \).

Claim: \( \forall \rho < r_0, a \in \mathcal{P} \), \( f \in \mathcal{O}(\Omega) \), the power series

\[
f(z) = \sum_{\alpha} \frac{f^{(\alpha)}(a)}{\alpha!} (z-a)^\alpha
\]

converges normally in \( \{ a \} + D_\rho \).

**Proof of Claim:** For \( z \in K \), \( \{ z \} + D_\rho \subset \Omega \Rightarrow \)

By Cauchy Estimates: \( |f^{(\alpha)}(z)| \leq \frac{\alpha! M^\rho}{\rho \alpha!} \), \( \forall x \in K \).

...
But \( f_{x} \in C_{0}(\Omega) \), \( |f_{x}(a_{j})| \leq \frac{e^{M_{a_{j}}}}{p_{a_{j}}} \), \( \forall a_{j} \in \Omega \)

\[
\Rightarrow \sum_{j} \frac{f_{x}(a_{j})(z-a_{j})^{k}}{k!} \text{ conv. normally in } \{a_{j}+D_{r} : \Omega \}
\]

To prove \( K_{x} \supset C \), suffices to prove \( d(K_{x}, C \setminus \Omega) > 0 \).

Since \( K_{x} \) is closed in \( \Omega \), but this follows easily from Claim. \( \{a_{j}+D_{r} : \Omega \} \) since \( \Omega \) is domain of holomorphy.

\((iii) \Rightarrow (i)\) : Construct \( f \in C_{0}(\Omega) \) as follows.

- Let \( L \subset \Omega \) be countable and dense \( (L = \Omega \cap \mathbb{Q}^{n} \text{ e.g.}) \).
- Let \( \{a_{j} : j \in L \} \) be sequence s.t. \( a_{j} \in L \) and every \( z \in L \) is s.t. \( a_{j} = z \) for infinitely many \( j \).
- Let \( \delta_{j} = \Delta_{\Omega}(a_{j}) \Rightarrow \{a_{j} + D_{\delta_{j}} \subset \Omega \} \) and contains points arbitrarily close to \( a_{x} \).
- Let \( K^{j} = \Omega \subset \Omega \) compact exhaustion, \( K^{j} \subset K^{j+1} \subset \cdots \subset C \Omega \) for every \( j \geq 0 \).

For every \( j \), \( \exists z^{j} \in \{a_{j} + D_{\delta_{j}} \}, z^{j} \notin K^{j} \).

By \( \text{def}^{\circ} \) of \( K^{j} \), \( \exists f_{j} \in C_{0}(\Omega) \) s.t. \( f_{j}(z^{j}) = 1 \), \( \sup |f_{j}| < \frac{1}{2^{j}} \).

\( z^{j} \notin K^{j} \Rightarrow f_{j} \notin K^{j+1} \) \( \forall j \). This product converges.
Consider: \[ f(z) = \prod_{j=1}^\infty (1 - f_j(z))^0 \] This product converges uniformly on compact to \( f(0, \Omega) \) w/ zeros only when \( f_j(z) = 1 \) some \( j \), \( b/c \sum_{j=1}^\infty i k_j \leq C \sum_{j=1}^\infty j 2^{-j} < \infty \), \( \forall \lambda \in \Omega \), Analysis Fact. \( \prod_{j=1}^\infty (1 - g_j) \) conv. unif. on compacts to \( f \) w/ 
\[ g = 0 \iff g_j = 1 \] if \( \sum_{j=1}^\infty |g_j| \) converges unif. on compacts.

Claim. \( \exists \lambda_1, \lambda_2 \in \Omega_1, f_2 \in \mathcal{O}(\Omega_2), f = f_2 \) in \( \Omega_1 \).

Contradiction. Suppose they exist.

Note that \( f(\lambda_1) = 0 \) and \( f_2(\lambda_1) = 0 \)
for all \( 1 \leq j \leq -1 \) by construction
(\( f \) is divisible by \( (1 - f_j(z))^j \))

Let \( \Omega_0 \) be component of \( \Omega \setminus \Omega_2 \) that contains \( \Omega_1 \), pick \( z^* \in \Omega_2 \), by contradiction there is subsequence \( z_{j_k} \)
\( z_{j_k} \to z^* \) as \( k \to \infty \),
But then \( z_{j_k} \to z^* \) as well, since \( f_2 = f \) in \( \Omega_0 \),
\( (f_2)_{z_k} (z_{j_k}) = 0 \), \( 1 \leq j_k \leq j_{k-1} \)
\[ \Rightarrow (f_2)_{z_k} (z^*) = 0, \forall \lambda \Rightarrow f_2 \equiv 0. \]
\[ \Rightarrow f \equiv 0 \] in component of \( \Omega_2 \) containing