VI. 1.1

Suppose $G$ is bounded open (not necessarily connected),
$f: \text{analytic on } G, \text{continuous on } \overline{G}, \text{non constant}$.

Want: "$ f \) has zero in $G$ or $\Rightarrow |f| \) assumes its minimum on $\partial G$."

If $f$ has zero in $\overline{G}$, then it's clear that it satisfies either 1 or 2).

2. Suppose $f$ is nowhere zero in $\overline{G}$. Then $1/f$ is analytic on $G$ and continuous on compact set $\overline{G}$. \[ \Rightarrow |1/f| \) assume its maximum, say $|1/f| = C \] for some $x \in \overline{G}$. If $x \in G = \bigcup G_a$ where $G_a$: connected component of $G$, hence open, then $x \in G_a$ for some $a$. Since $1/f: G_a \to C \) assume its maximum, by MMP $1/f$ is constant on $G_a$. By continuity of $1/f$ on $\overline{G}$, $1/f$ is constant on $\partial G_a \subset \partial G$ \[ \Rightarrow |1/f| = |1/f(y)| = C \] for some $y \in \partial G$.

(by topological reason)

= $|f| \) assume its maximum on $\partial G$, equivalently \[ \Rightarrow \] holds \[ \text{(MMP)} \]
Suppose \( \exists \) analytic \( f: \mathbb{D} \to \mathbb{D} \) s.t. \( f\left(\frac{1}{3}\right) = \frac{3}{4} \), \( f'(\frac{1}{3}) = \frac{2}{3} \).

Define \( h: \mathbb{D} \to \mathbb{D} \) by \( h = \psi_{\frac{3}{4}} \circ f \circ \psi_{-\frac{1}{3}} = \psi_{\frac{3}{4}} \circ f \circ \psi_{-\frac{1}{3}} \).

Then \( h(0) = 0 \), hence satisfies condition for Schwartz's lemma.

In particular, we have \( |h(0)| \leq 1 \). By chain rule,

\[
|h'(0)| = |\psi_{\frac{3}{4}}'(\frac{3}{4})| \cdot |f'(\frac{1}{3})| \cdot |\psi_{-\frac{1}{3}}'(0)| = \frac{1}{1 - (\frac{3}{4})^2} \cdot f'(\frac{1}{3}) \cdot (1 - (\frac{3}{4})^2) \\
\Rightarrow |f'(\frac{1}{3})| = \frac{1 - \frac{9}{16}}{1 - \frac{1}{4}} \cdot |h'(0)| \leq \frac{7}{12}.
\]

On the other hand, \( f'(\frac{1}{3}) \) was given by \( \frac{2}{3} > \frac{7}{12} \).

\( \therefore \) such analytic function \( f \)
Suppose $f$ is analytic in some region containing $\overline{B(0)}$, and $|f(z)| = 1 \quad \forall z \in \overline{D}$.

**Case 1:** $f$ has no zero in $\overline{D}$.

We claim that $f$ is constant function. Otherwise, by VI.1.1,

If $f$ should assume minimum on $\partial \overline{D} = \{ z : |z| = 1 \}$, then $|f(z)| \geq 1 \quad \forall z \in \overline{D}$.

But also MMP implies $|f(z)| \leq 1 \quad \forall z \in \overline{D}$. Hence $f$ is not constant function, $|f(z)| = 1 \quad \forall z \in \overline{D}$. But this cannot happen by open mapping theorem. Thus $f$ is constant.

**Case 2:** General case.

If $f$ has infinitely many zeros in $\overline{D}$, then by compactness, it admits some accumulated point hence $f \equiv 0$ by identity theorem. (Here we used $f$ is analytic on the neighborhood of $\overline{D}$).

Then $h(z) := \frac{f(z)}{\prod_{i=1}^{\infty} p_i(z)}$ satisfies all the condition for case 1.

(Most importantly, $p_i(z)$ has unique simple zero at $z_i$ and

$|p_i(z)| = 1 \quad \forall |z| = 1$)

So $h(z) \equiv C$, equivalently $f(z) = C \cdot \prod_{i=1}^{n} p_i(z)$.

Therefore, $f(z) = C \cdot \prod_{i=1}^{n} p_i(z)$ for some $C \in \mathbb{C}$, $z \in \overline{D}$.