VI.3.6 \[ f: B(c,R) \to \mathbb{C} \text{ analytic, not constant. Define } I(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta \text{ for } r < R. \]

Consider \( A := \{ r : 0 \leq r < R, \exists \theta \text{ s.t. } f(re^{i\theta}) = 0 \} \subseteq \{0, \ldots, R\} \).

Then \( A \) is discrete subset in \( (0,R) \) if \( c \) otherwise, \( f \) has infinitely many zero in closed ball which implies \( f = 0 \) by \( r \notin A \).

For each \( r \in (0,R) \setminus A \), define continuous function \( \psi_r: [0,2\pi] \to \mathbb{S} \) by \( \psi_r(\theta) := \frac{|f(re^{i\theta})|}{f(re^{i\theta})} \).

Define \( F_r(z) := \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) \psi_r(\theta) \, d\theta \) on \( B(c,R) \). Then \( F \) is clearly analytic.

Also define \( M_r(r') := \sup_{\theta} |F_r(e^{i\theta})| \). Note following two facts:

1. \( M_r(r') \leq I(r') \quad \forall r' \)

\[ \quad (\because \text{ for } |z| = r', \quad |F_r(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| \psi_r(\theta) \, d\theta = I(r'), \quad \text{take supremum}) \]

2. \( M_r(r) = I(r) \)

\[ \quad (\because F(0) = I(0) \text{ by def } \Rightarrow M_r(0) = I(0)) \]

Pick any \( 0 < r_1 < r < r_2 < R \) where \( r \notin A \). Then, we have following:

\[ (\log r_2 - \log r_1) \log I(r) = (\log r_2 - \log r_1) \log M_r(r) \leq (\log r - \log r_1) \log M_r(r_1) + (\log r_2 - \log r) \log M_r(r) \]

\[ r \notin A \quad (\text{F(z) analytic}) \]

\[ \leq (\log r - \log r_1) \log I(r_1) + (\log r_2 - \log r_1) \log I(r) \]

But since \( A \) is discrete subset of \((0,R)\), for \( r \notin A \), we can take limit of above inequality. \( \log I(r) \) is logr convex w.r.t.
Now, let's prove monotonic increasing property of \( I(r) \). Suppose not. Then \( \exists \epsilon < \delta \) s.t. \( I(r) > I(r_\epsilon) \). By continuity of \( I(r) \) \( (\text{check this!}) \), \( \exists \delta > 0 \) s.t. if \( |r - r_\epsilon| < \delta \) then \( I(r) > I(r_\epsilon) \). By discreteness of \( A \subseteq (0, R) \), \( \exists \epsilon \in (0, R) \backslash A \) s.t. \( r_\epsilon - \delta < r < r_\epsilon \).

Then,

1. \[ M_r(r) = I(r) > I(r_\epsilon) \]
2. \[ M_r(r) \leq I(r_\epsilon) \]
3. \[ M_r(r_\epsilon) \leq I(r_\epsilon) \]

But this contradicts that \( M_r \) is monotonic increasing by MMP: \( I(r) \) is monotonically increasing.

To prove \( I(r) \) is actually strict, again suppose not. Then \( \exists \epsilon < \delta \) s.t. \( I(r) = I(r_\epsilon) \).

By monotonicity above, \( I(r) \) is constant in interval \( [r, r_\epsilon] \). By shrinking \( [r, r_\epsilon] \), if necessary, we may assume that \( [r, r_\epsilon] \cap A = \emptyset \).

Claim: \( \exists \epsilon \in (r, r_\epsilon) \) s.t. \( M_r \) is not constant.

• Suppose \( M_r \) is constant \( \forall \epsilon \in [r, r_\epsilon) \). Then,

\[
M_r(r_\epsilon) = \frac{1}{2\pi} \int_0^{2\pi} f(r e^{i\theta}) e_r^{i\theta} r_\epsilon \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(r e^{i\theta}) e_r(0, -\tau) \, d\theta
\]

\[
M_r(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r e^{i\theta}) \epsilon_r(0, \tau) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} |f(r e^{i\theta})| \, d\theta.
\]

Since integrand \( (*) \) has modulus \( |f(r e^{i\theta})| \), for integral to be same, we need to have \( f(r e^{i\theta}) \epsilon_r(0, -\tau) = |f(r e^{i\theta})| = \epsilon_r(0, \tau) f(r e^{i\theta}) \) \( \forall \tau \epsilon \).

\[ \epsilon_r(0, -\tau) = \epsilon_r(0, \tau) \Rightarrow \epsilon_r \text{ is constant } \forall \tau \epsilon \]

It is very special kind of function which sends each circle to line. (i.e. For fixed \( r \), angular \( \epsilon_r(0, \tau) \) is constant)

We will argue briefly why such function should be constant. Therefore claim is proven.
\[ f : \mathbb{C} \rightarrow \mathbb{C} \text{ analytic s.t. } \forall r \in (r_1, r_2), \exists \theta \text{ s.t. } \text{Angular}(f(re^{i\theta})) = 0 \]

\[ \Rightarrow f \text{ is constant.} \]

Sketch of proof: Since \( f \) sends circle to line of constant angle, by conformality of analytic function, \( f \) sends angle to circle. i.e. we have function \( r : [0, 2\pi] \rightarrow [0, \infty) \) s.t.

\[ \text{Angle}(z) = \theta \Rightarrow |f(z)| = r(\theta). \]

Clearly, function \( r \) is constant, hence attain maximum. Say \( r(\theta_0) \) is the maximum. Then \( |f(re^{i\theta_0})| : \text{maximum} \)

for any \( r_1 < r < r_2 \). By MMP, \( f \) is constant.
\[ \text{VII.5} \quad \text{Supp } f_n \to f \text{ in } C(G, \mathbb{R}), \ z_n \to z \text{ in } G. \]

Pick any \( \delta > 0 \). \( \text{Want: } d(f_n(z), f(z)) < \delta \text{ for } n \to \infty. \)

Pick any \( K^+ \subseteq G \) s.t. \( z \in \text{int}(K) \). Then \( z_n \notin K \) for \( n \geq N_1 \) by convergence.

For given \( K^+ \), \( \delta > 0 \), \( \exists \varepsilon > 0 \) s.t. \( \rho(q, h) < \varepsilon \implies \rho_K(f_q, f_h) < \delta. \)

By convergence \( f_n \to f \), \( \rho(f_q, f_h) < \varepsilon \) for \( n \geq N_2 \), hence \( \rho_K(f_n, f) < \delta. \)

In particular, \( d(f_n(z), f(z)) < \delta \) if \( n \geq \max\{N_1, N_2\} \) (\( = z \in K \)).

By continuity of \( f \), \( d(f(z), f(z)) < \delta \) if \( n \geq N_3. \)

\( \implies \) If \( n \geq \max\{N_1, N_2, N_3\} \), then \( d(f(z), f(z)) \leq d(f(z), f(z_n)) + d(f(z_n), f(z)) < 2\delta. \)

\( = f(z_n) \to f(z) \text{ in } G. \]

\[ \text{VII.6} \quad \text{(Darboux's theorem)} \quad \text{Supp } f_n \in C(G, \mathbb{R}), f_n \to f_n, \ \exists \varepsilon \in C(G, \mathbb{R}) \text{ s.t. } \lim_{n \to \infty} f_n = f. \]

\( \text{Want: } f_n \to f \text{ in } C(G, \mathbb{R}), \text{ equivalently, } \forall K^+, \rho_K(f_n, f) \to 0 \text{ as } n \to \infty. \)

By using compactness argument, this is also equivalent to \( \text{locally uniform convergence}. \)

i.e., \( \forall z \in G, \exists \text{ open nbd } U \ni z \text{ s.t. } \rho_U(f_n, f) \to 0 \text{ as } n \to \infty. \)

\( \text{Pick any } z \in G. \text{ Let } \varepsilon > 0. \text{ By continuity of } f, \exists U \ni z \text{ s.t. } \forall w \in U \implies |f(z) - f_w| < \varepsilon. \)

By \( f_n(z) \to f(z), |f(z) - f_n(z)| < \varepsilon_2 \text{ for some } n \geq N \). Also, by continuity of \( f_N, \)

\( \exists \text{ open set } V : z \in V \subseteq U \text{ s.t. } \forall w \in V \implies |f_N(z) - f_N(w)| < \varepsilon_2. \)

\( \implies \forall w \in V, |f_N(w) - f(w)| \leq |f_N(z) - f_N(w)| + |f_n(z) - f(z)| + |f(z) - f(w)| \leq \varepsilon_2 + \varepsilon_2 + \varepsilon = 2\varepsilon. \)

\( \implies \text{by monotonicity of } |f_n|, \text{ we know } \rho_V(f_n, f) \text{ decreases only. } \)

This proves that \( \rho_V(f_n, f) \to 0 \text{ as } n \to \infty \implies f_n \to f \text{ in } C(G, \mathbb{R}). \)