Hints for HW in Ch VIII 1.6

\( K \subseteq G \text{ compact, } \hat{K}_G := \{ z \in G : |f(z)| \leq \sup_{K} |f|, \forall f \in H(G) \} \).

Recall: \( \hat{K} := \{ z \in \mathbb{C} : |p(z)| \leq \sup_{K} |p|, \forall \text{ polynomials } p(z) \} \).

Since each polynomial \( p \) is also in \( H(G) \Rightarrow \hat{K}_G \subseteq \hat{K} \).

(a) If \( G \) is connected, polynomials approximate all \( f \in H(G) \) by Runge. Use this to prove \( \hat{K}_G = \hat{K} \).

(b) Let \( a \in \hat{K}_G \), \( b \in \mathbb{C} \setminus G \) s.t. \( |a - b| = d(\hat{K}_G, \mathbb{C} \setminus G) \).

Since \( b \in \mathbb{C} \setminus G \Rightarrow \frac{1}{z - b} \) is analytic in \( G \).

\( a \in \hat{K}_G \Rightarrow |a - b| \leq \sup_{K} \left| \frac{1}{z - b} \right| \). Use this to prove \( d(K, \mathbb{C} \setminus G) \leq d(\hat{K}_G, \mathbb{C} \setminus G) \).

(c) Any half space can be described by an equation \( \Re(e^{cz}) \leq y \), \( c \in \mathbb{C} \), \( y \in \mathbb{R} \). Now, for any \( z \in G \), the function \( e^{cz} \) is analytic in \( G \), so if \( z \in \hat{K}_G \), we have

\[ |e^{cz}| = \sup_{z \in K} |e^{cz}|. \]

Use this to prove (c). (enough to use half spaces to describe the convex hull?)

(d) Skip, as hint refers to #4 which was not assigned.

(e) Skip. Also uses partly #4. However, you should be able to prove, using Max Mod Principle, that if \( D \)
is a odd component of $G \setminus K$ st. $D$ does not meet $E_a$, then $D \subseteq R_a$. Note that in this case $2D \subseteq K$ ...