Ch VI. Max Modulus Thm
§1 Max (Modulus) principle

The maximum principle is a fundamental tool in complex analysis/geometry. Here we discuss them in the context of analytic (holomorphic) functions, as we already have the tools for this. More generally, there are max principles for harmonic and subharmonic functions, but this discussion requires more preliminaries (but we may get to this).

Max Modulus Thm - I. Let \( G \subseteq \mathbb{C} \) be a region (i.e. open connected), and \( f(z) \) an analytic function in \( G \). If \( |f(z)| \) achieves a max at a point \( \alpha \in G \), then \( f(z) \) is constant.

Proof. The Open Mapping Thm. If \( f(z) \) not constant, then \( f(G) \) is open. Let \( \alpha = f(a) \) and assume

\[ |z| \leq |f(z)| \quad \text{for} \quad z \in G. \]

I.e. \( \alpha \in f(G) \) open \( \Rightarrow \exists \overline{B(\alpha, r)} \subseteq f(G) \), but

\[ 0 \]
By construction, $\beta \in f(\alpha)$ so $\beta = f(\beta)$ for some $\beta \in \Theta$. But clearly $|\beta| > |\alpha|$, which is a contradiction. \(\square\)

Ex. $\Gamma$ must be connected. Suppose $\Gamma = \Theta \cup B(2, r), r < 1$

Consider $f(z) = \begin{cases} z, & z \in \Theta \\ 2, & z \in B(2, r) \end{cases}$

Note: $|f(z)|$ achieves a max at e.g. $z = 2 \in \Gamma$, but $f(z)$ is not constant on $\Gamma$.

Rem. Connectedness is not a major restriction. We can apply MMT-I to each component.

Two useful corollaries of MMT-I.

1. MMT-II. Let $\Gamma$ be open in $\mathbb{C}$, $f(\Gamma)$ analytic in $\Gamma$ and continuous on $\overline{\Gamma}$. Then $\max |f(z)| = \max |f(\Gamma)|$.

Def. Note $\overline{\Gamma} \subseteq \mathbb{C}$ is compact. Let $a \in \overline{\Gamma}$ be s.t. $|f(a)| = \max |f(z)|$. \(\square\)
If \( a \in G \), then \( a \in G_1 \), where \( G_1 \subseteq G \) is a component. Thus, we can apply MMT-I to conclude that \( f(z) \) is constant on \( G_1 \Rightarrow \max |f(z)| = \max |f(z)| \Rightarrow \) conclusion of MMT-II.

If \( a \notin G \Rightarrow a \notin G \Rightarrow \) conclusion as well.

2. For \( G \subseteq \mathbb{C} \) open, let \( a \in \partial_{\infty} G \) and set

\[
(G-) \limsup_{z \to a} |f(z)| := \lim \sup_{r \to 0, r > 0} \{|f(z)| : z \in G \cap B(a, r)\}
\]

If \( a = \infty \) (only if \( G \) unbounded), then \( B(\infty, r) \) means ball in the F-S metric on \( \mathbb{C} \), i.e.,

\[
B(\infty, r) \setminus \{\infty\} := \{z \in \mathbb{C} : \frac{2}{(1+|z|^2)^{1/2}} < r \}
\]

\[
= \{z \in \mathbb{C} : |z|^2 > \frac{4}{r^2} - 1\}
\]

Rem.: Could use \( B(\infty, r) \setminus \{\infty\} = \{z \in \mathbb{C} : |z| > \frac{1}{r} \} \) in def of \( \limsup_{z \to \infty} |f(z)| \).

\[\underline{\text{MMT-III}}\]

Let \( G \subseteq \mathbb{C} \) be open, \( f(z) \) analytic in \( G \). Assume

\[
\limsup_{z \to a} |f(z)| \leq M, \quad \forall a \in \partial_{\infty} G
\]
Then $|f(z)| \leq M$ in $G$.

Proof. Pick $\varepsilon > 0$, consider $E = \{z \in G : |f(z)| > M + \varepsilon\}$. 
$E$ is clearly open since $|f(z)|$ is cont. in $G$. 
Want: $E = \emptyset$.
Assume not. We claim $E \subseteq G$ and $E$ is bdd. 
Clearly, $E \subseteq G$. So if the claim is not true, then $\exists a \in \partial G$ s.t. $a \in \partial E$, but 
$E$-limsup $|f(z)| \leq G$-limsup $|f(z)| \leq M$ \(z \rightarrow a\) 
which cannot be since $|f(z)| > M + \varepsilon$ in $E$. 
So $E \subseteq G$ is open & bdd, and $E \subseteq G$ 
$|f(z)|$ cont. in $E$, 

MMT-II applies $\implies \max_{E} |f(z)| = \max_{\partial E} |f(z)| = M + \varepsilon$ 

But since $E \neq \emptyset$, $\max_{E} |f(z)| > M + \varepsilon$. 

This is a contradiction $\implies E = \emptyset$. 

This proves MMT-III. $\Box$