Let \( G \subseteq \mathbb{C} \) be region, \( f \in M(G) \). We define a non-negative function \( m_f \in C(G, \mathbb{R}) \) as follows:

1. **If \( a \in G \) is not a pole:**
   \[
m_f(a) = \frac{2 |f'(a)|}{1 + |f(a)|^2}.
   \]

2. **If \( a \in G \) is a pole:**
   \[
m_f(a) = \lim_{z \to a} m_f(z), \quad \text{(which exist by b)}
   \]
   Note: Let \( f(z) = \frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{z-a} + f_0(z) \), then
   \[
f'(z) = -\frac{(m A_m)}{(z-a)^{m+1}} + \cdots + \frac{A_1}{(z-a)^2} + f_0'(z)
   \]
   \[
   \Rightarrow 2 |f'(z)|^2 = \frac{|(z-a)^{2m-2(m+1)} A_m + O(z-a)|}{|A_m|^2 + O(z-a)}
   \]
   \[
   \Rightarrow \begin{cases} 
   0 & \text{if } 2m-m-1 = m-1 \geq 1; \text{i.e., } m \geq 2 \\
   \frac{2}{|A_1|} & m = 1
   \end{cases}
   \]

Remark: Fubini-Study metric \( d_\mu(z_1, z_2) \) comes from Riemannian metric
\[
ds^2 = \frac{4 |dz|^2}{(1 + |z|^2)^2}.
\]
If \( f: C_0 \to C_0 \) smooth map then \( f^* ds^2 = \mu_f ds^2 \).

Thus 1. A family \( f \in M(G) \) is normal in \( C(G, \mathbb{C}) \).
(or eq. in $W(\mu)\subset C^1$, which is complete) \(\Rightarrow\)
\[
\{\mu_f: f \in F\} \text{ is locally bad.}
\]

\textbf{Pf:} Since \(\Omega\) is compact, (i) in AA-Thm 1 is always true, so normality in \(C(\Omega, \mathbb{C})\) \(\Leftrightarrow\) Equicont. at every \(a \in \Omega\).

1. \(\Leftrightarrow\) in Thm 1. Fix \(a \in \Omega, \varepsilon > 0\). First, consider \(K = B(a, \varepsilon) \subset \mathbb{C}\) and let \(M > 0\) be s.t.
\[
\mu_f(z) \leq M, \forall f \in F, z \in K.
\]

Pick \(f \in F\) and \(z \in K\). Cases:

1. \(a, z\) both not poles of \(f\). Let \(P\) polygonal path from \(a\) to \(z\) s.t. no pole of \(f\) lies on \(P\). \((P\) may depend on \(f\))

\textbf{WLOG:} \((a)\ l(P) = \text{length of } P \leq 2|z-a|

For any partition \(a = w_0, \ldots, w_n = z, w_{n+1} = \ldots, w_n = z\) of \(P\).
\[
d(f(a), f(z)) \leq \sum_{k=1}^{n} d(f(w_{k-1}), f(w_k)) = 2 \sum_{k=1}^{n} \frac{|f(w_k) - f(w_{k-1})|}{\left(\left(1 + |f(w_k)|^2\right) \left(1 + |f(w_{k-1})|^2\right)\right)^{1/2}}
\]
\[
d = d_\infty
\]
\[
\leq \sum_{k=1}^{n} \frac{|f'(z_k)|}{M} |w_k - w_{k-1}|, \quad \exists z_k \in [w_{k-1}, w_k]
\]
\[ = 2 \sum_{h=1}^{n} \mathcal{E}(s_h) \cdot \frac{1 + |P(s_h)|^2}{(1 + |P(s_h)|^2)^{1/2} (1 + |P(w_h)|^2)^{1/2} \gamma(s_h, w_h, s_h)} \]

\[ \leq 2M \sum_{h=1}^{n} \gamma(s_{h-1}, w_h, s_h) |w_h - w_{h-1}|, \tag{*} \]

where \( \gamma(s, t, u) = \frac{1 + |P(t)|^2}{(1 + |P(s)|^2)^{1/2} (1 + |P(t)|^2)^{1/2}} \)

**Note:** \( \gamma : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous and \( \gamma(s, s, s) = 1 \), \( \forall s \in \mathbb{R} \). Since \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) is compact, \( \gamma \) is uniformly continuous \( \Rightarrow \exists \alpha > 0 \) s.t. \( |s-t| < \alpha, \ |s-u| < \alpha \Rightarrow \gamma(s, t, u) \leq 2 \).

Choose partition \( w_0, \ldots, w_n \) above s.t. \( |w_h - w_{h-1}| < \alpha \)

\[ \Rightarrow |s_h - w_{h-1}| < \alpha \Rightarrow \gamma(s_{h-1}, w_h, s_h) \leq 2. \]

\( (*) \Rightarrow d(P(a), P(z)) \leq 4M \sum_{h=1}^{n} |w_h - w_{h-1}| = 4M L(P) \]

\[ \leq 8M |z - a|. \]

\( (2) \) a not a pole, \( z \) a pole (or vice versa).

For a partition as above:

\begin{align*}
&\text{New Section 1 Page 3}
\end{align*}
\[ d(f(a), f(z)) \leq \sum_{n=1}^{\infty} d(f(w_{n-1}), f(w_n)) + d(f(w_n), \infty) \leq 8M |w_{n-1} - a| + d(f(w_n), \infty) \]

Since \( d(f(w_n), \infty) \to 0 \) as \( w_n \to \infty \) and \( |w_{n-1} - a| \to |z - a| \), we conclude that \( d(f(a), f(z)) \leq 8M |z - a| \) also in this case.

(3) \( a, z \) both poles. Similar argument to (2).

We conclude that for any \( z \in K \), \( f \in F \), we have \( d(f(a), f(z)) \leq 8M |z - a| \Rightarrow \text{equicontinuity at } a \).

This completes proof of \( \Leftarrow \) in Thm 1.

The proof of \( \Rightarrow \) in Thm 1 is HW*4 problem. \( \Box \)