For \( G \subseteq \mathbb{C} \) region, define for \( f \in M(G) \subseteq C(G, C_0) \) the continuous function \( \mu_f : G \to \mathbb{R} \) by

\[
\mu_f(z) = \begin{cases} 
\frac{2 \Im(f(z))}{1 + |f(z)|^2} & z \text{ not a pole} \\
\frac{2}{\Re f(z)} & z \text{ pole of order 1} \\
0 & z \text{ pole of order } \geq 2.
\end{cases}
\]

**Theorem 1.** A family \( F \subseteq M(G) \) is normal \( \implies \) 
\( \{ \mu_f : f \in F \} \) locally hold.

**Proof.** \( \Leftarrow \): Lecture 11.

\( \Rightarrow \):

Claim: \( f_n \to f \text{ in } M(G) \implies \mu_{f_n} \to \mu_f \text{ in } C(G) \)

**Proof:** Sufficient to show \( \forall \alpha \in G \exists B(\alpha, r) \subseteq G, s.t., \mu_{f_n} \to \mu_f \text{ uniformly in } B(\alpha, r). \)

\( 1 \) \( \alpha \) is not a pole of \( f \). Then, \( F(B(\alpha, r)), M > 0 \) s.t. 

\[ |f_n(z)|, |f(z)| \leq M \quad \text{ and } \quad f_n \to f \text{ uniformly in } B(\alpha, r). \]

Note that \( w(\omega z) = \frac{\omega w_1}{1 + \omega w_2}z \) is unif. cont. in \( \{ w_1, w_2 : 1 \leq R \}

Since \( f_n \to f \), \( f_n \to f \text{ uniformly in } B(\alpha, r) \Rightarrow \)

12.1. \( f_n \rightrightarrows \frac{f_n}{f} \text{ uniformly in } B(\alpha, r). \)

\[ 1 + \frac{1}{n^2} \to \mu_f = \frac{1}{1 + \frac{1}{n^2}} \begin{array}{c}
(2) \text{ a } \varphi \text{ pole of } f. \text{ Consider } \frac{1}{f} \text{ and } \frac{1}{f_n}, \text{ and note } \mu_{\frac{1}{f}} = \mu_f. \text{ By above, } \mu_f = \mu_{\frac{1}{f_n}} \to \mu_{\frac{1}{f_n}} = \mu_f. \end{array} \]

To prove \( \implies \) in Thm 1, suppose \( f \) normal, but \( \{\mu_{\frac{1}{f_n}}: \frac{1}{f_n} \in E\} \) not locally closed. Then, \( f \in \mathcal{C}G \), \( z_n \to a \), then \( w \) \( \mu_{\frac{1}{f_n}}(z_n) \to h \). But \( f \) normal \( \implies \exists \text{ subseq. } \{f_n\} \) s.t. \( f_n \to f \) in \( \mathcal{C}(\mathcal{G}, \mathcal{C}o) \). Suppose first \( f \neq 0 \), i.e., \( f \in \mathcal{MC}(\mathcal{G}) \). Then, by Claim, \( \mu_{f_n} \to \mu_f \) in \( \mathcal{C}(\mathcal{G}) \).

Take \( K = B(a, r) \subseteq \mathcal{G} \). \( \mu_{f_n} \to \mu_f \) uniform on \( K \) \( \implies \mu_{f_n} \leq M \) \( \forall n \). But \( \mu_{f_n}(z_n) \to \infty \) and \( z_n \to a \). Contradiction. If \( f = 0 \), then consider \( g_n = \frac{1}{f_n} \to g = 0 \).

We again get a contradiction since \( \mu_{g_n} = \mu_{g_n} \).