Lecture 14

- Finish proof of RMT (Lecture 13 notes).

Recall: If \( p \in H(G) \) and has zeros at \( a_1, \ldots, a_n \) of orders \( m_1, \ldots, m_n \), then
\[
f(z) = \left( \prod_{k=1}^{n} (z-a_k)^{m_k} \right) f_0(z), \quad f_0 \in H(G).
\]

- It is convenient, to list the zeros repeated according to multiplicities:
\[
\underbrace{a_1, \ldots, a_1}_{m_1}, \overbrace{a_2, \ldots, a_2}^{m_2}, \ldots, \underbrace{a_n, \ldots, a_n}_{m_n}
\]

We write \( \tilde{a}_1, \ldots, \tilde{a}_N \) \( \rightarrow \). Then
\[
f(z) = \left( \prod_{k=1}^{N} (z-\tilde{a}_k) \right) f_0(z)
\]

- \( f(z) \) can have an infinite, countable \( \# \) of zeros, r.a.m., \( \{a_n\} \sim \infty \). (No accumulation points in \( G \), though).

Q: Can we factor
\[
f(z) = \left( \prod_{k=1}^{\infty} (z-a_k) \right) f_0(z)
\]

What is \( \prod_{n=1}^{\infty} Z_n \), \( \{Z_n\} \) seq. of points in \( G \)?

Well \( \prod_{n=1}^{\infty} Z_n = \lim_{N \to \infty} \prod_{n=1}^{N} Z_n \) if the sequence \( \forall N \in \mathbb{N} = \prod_{n=1}^{N} Z_n \)
Well, \( \prod_{n=1}^{N} z_n = \lim_{N \to \infty} \prod_{h=1}^{N} z_{2h} \) if the sequence \( WN = \lim_{N \to \infty} \prod_{h=1}^{N} z_{2h} \) converges. We say \( \prod_{h=1}^{N} z_{2h} \) converges to \( W = \lim_{N \to \infty} \prod_{h=1}^{N} z_{2h} \).

**Ex:** \( z_n = \frac{1}{2} \Rightarrow WN = \prod_{h=1}^{N} \frac{1}{2} = \frac{1}{2^N} \rightarrow 0. \)

Not good: A product (infinite) of non-zero numbers is zero.

**Prop 1:** Suppose \( \{z_n\} \) is seq. of complex numbers w/ \( \text{Re} z_n > 0 \)

Then! \( \prod_{h=1}^{N} z_{2h} \) converges to \( W \neq 0. \)

\[ \iff \sum_{h=1}^{\infty} \text{Log } z_{2h} \text{ converges.} \quad (\text{Log, principal branch}). \]

**Proof:** \( \Leftarrow \). Let \( 3N = \sum_{h=1}^{N} \text{Log } z_{2h} \). We have

\[ WN = \prod_{h=1}^{N} z_{2h} = e^{\sum_{h=1}^{N} \text{Log } z_{2h}} = e^{3N}. \]

By assumption, \( \exists z \in \mathbb{C} \) s.t. \( 3N \rightarrow \mathbb{C} \). By continuity

\[ WN = e^{3N} \rightarrow e^3. \] I.e. \( \prod_{h=1}^{N} z_{2h} \) converges to \( W = e^3 \neq 0. \)

\[ \Rightarrow WN \rightarrow W = re^{i\theta}, \; r \neq 0, \; \theta \in (-\pi, \pi]. \]
Define branch of
\[ \log z = \log |z| + i \theta \]
\[ \theta \in (\theta - \pi, \theta + \pi] \]

\[ W_N = e^{l(w_N)} \text{ and } w_N = e^{2N} \implies 3_N = l(w_N) + 2\pi i k_N, \]
\[ k_N \in \mathbb{Z}. \]

For \( N \) large enough, \[ l(w_N) = l(w_{N-1}) = \log \frac{w_N}{w_{N-1}} = \log z_N \]

\[ \implies \log z_N = 3_N - 3_{N-1} = l(w_N) - l(w_{N-1}) + 2\pi i (k_N - k_{N-1}) \]

\[ = \log \left( \frac{w_N}{w_{N-1}} \right) + 2\pi i (k_N - k_{N-1}) = \log z_N + 2\pi i (k_N - k_{N-1}) \]

\[ \implies k_N - k_{N-1} = 0. \text{ Thus, } 3_N = l(w_N) \]

Since \( w_N \to w \), \[ 3_N = l(w_N) \to l(w) = 3. \]