Lecture 17

Recall: Elementary Factors

\[ E_0(z) = (1-z), \quad E_p(z) = (1-z) z^p + \cdots + z^p, \quad p \in \mathbb{N}, \quad \text{entire} \]

Note: If \( a \in \mathbb{C} \), then \( E_p(z/a) \) has simple zero at \( z = a \), no other zero.

Lemma: \( |E_p(z) - 1| \leq |z|^p+1 \), \( \forall |z| \leq \frac{1}{2} \).

Pt. See Lecture 16, \( \Phi \).

Recall: Let \( G \subseteq \mathbb{C} \) region, \( \{a_n\}_{n=1}^{\infty} \) seq. in \( G \). \( \{a_n\} \) is a zero sequence if \( \{a_n\} \) has no limit point in \( G \).

Rem. \( \forall a \in \mathbb{C} \) is a limit point of \( \{a_n\} \) if \( \exists \) subseq. \( \{a_{n_k}\}_{k=1}^{\infty} \) s.t. \( a_{n_k} \to a \). Thus, if \( \{a_n\} \) has no limit point in \( G \), then any point \( a \in \mathbb{C} \) occurs only finitely many times in \( \{a_n\} \) and any limit point \( a = \lim_{n \to \infty} a_n \) belongs to \( \partial G \).

2) If \( G = \mathbb{C} \), then \( \{a_n\}_{n=1}^{\infty} \) is zero seq. \( \implies \lim_{n \to \infty} |a_n| = 0 \).

Thus, let \( \{a_n\}_{n=1}^{\infty} \) be zero sequence in \( \mathbb{C} \) (\( \lim_{n \to \infty} |a_n| = 0 \)), \( a_n \neq 0 \), and let \( \{E_{p_n}\}_{p=1}^{\infty} \) be seq. in \( \mathbb{N} \) s.t.

\[ \sum_{h=1}^{\infty} \left( \frac{1}{|a_n|} \right)^{p_h+1} < \infty, \quad \forall r > 0. \]

Then, \( f(z) = \prod_{h=1}^{\infty} E_{p_h}(z/|a_n|) \) converges in \( \mathbb{H}(S) \)
Then, \( \Phi(z) = \prod_{n=1}^{\infty} E_{P_n}(\frac{z}{\lambda_n}) \) converges in \( H(\mathbb{C}) \) and the seq. of zeros of \( \Phi \) (R.A.M.) coincides with \( \{\lambda_n\} \).

Moreover, \( P_n = n-1 \) always holds.

**Pf.** By previous, Thus, suffices to show that for each compact \( K \subset \mathbb{C} \),
\[
\sum_{n=1}^{\infty} (E_{P_n}(\frac{z}{\lambda_n})-1) \text{ conv. abs. unif. in } K.
\]

Pick \( K \subset \mathbb{C} \) compact. Pick \( r > 0 \) s.t. \( K \subseteq \{ |z| \leq r \} \).

Since \( \text{Im}(\lambda_n) \rightarrow 0 \) \( \exists N \text{ s.t. } |\frac{z}{\lambda_n}| \leq \frac{1}{2} \) for \( n \geq N \), \( z \in K \)

By Lemma 1, \( |E_{P_n}(\frac{z}{\lambda_n})-1| \leq \left| \frac{z}{\lambda_n} \right|^{P_{n+1}} \leq \left( \frac{r}{\lambda_n} \right)^{P_{n+1}} \), \( z \in K \)

Since \( \sum_{n=1}^{\infty} \left( \frac{r}{\lambda_n} \right)^{P_{n+1}} < \infty \) \( \Rightarrow \sum_{n=1}^{\infty} (E_{P_n}(\frac{z}{\lambda_n})-1) \text{ conv. abs. unif. on } K. \)

Next, note by above that for each fixed \( r \), \( \frac{r}{|\lambda_n|} \leq \frac{1}{2} \)

for \( n \geq N \Rightarrow w/ P_n = n-1 \) we get \( \left( \frac{r}{\lambda_n} \right)^{P_{n+1}} \leq \left( \frac{1}{2} \right)^n \), \( n \geq N. \)

\[
\Rightarrow \sum_{n=1}^{\infty} \left( \frac{r}{\lambda_n} \right)^{P_{n+1}} < \infty. \quad \Box
\]

**Ex.** Let \( a_n = n \in \mathbb{R} \). Then, \( \left( \frac{r}{a_n} \right)^{P_{n+1}} = \left( \frac{r}{n} \right)^{P_{n+1}}. \)

We can take \( P_n = 1! \)
\[
\sum_{n=1}^{\infty} \left( \frac{r}{n} \right)^2 = r^2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \forall r > 0.
\]
Weierstrass Factorization

Thus, let \( \Phi(z) \) be entire w/zero of
Weierstrass Factorization Theorem. Let \( f(z) \) be entire w/ zero of
multiple \( m \) at \( z = 0 \). Let \( \{a_n\}_{n=0}^\infty \) be seq. of remaining zeros
(R.A.M.). Then, \( \exists g(z) \) entire and \( \{b_n\}_{n=0}^\infty \) s.t.
\[
f(z) = z^m e^{g(z)} \prod_{n=1}^\infty E_n \left( \frac{z}{a_n} \right).
\]

Proof. By Thm 1, \( \exists \{b_n\} \) s.t. \( f(z) = z^m \prod_{n=0}^\infty E_n \left( \frac{z}{a_n} \right) \) convs.
to entire function w/ exactly same seq. of zeros
(R.A.M) as \( f(z) \). Thus, \( h(z) = \frac{f(z)}{f(z)} \) is
entire w/ no zeros. Since \( f \) is s.c., \( \exists g(z) \) s.t.
\[
h(z) = e^{g(z)}.
\]

Ex. \( f(z) = \sin \pi z \). The zeros of \( f \) are simple
zeros at \( z = n \in \mathbb{Z} = \{-\ldots, -1, 0, 1, \ldots \} \).

Let \( \sum' = \sum_{n=-\infty}^{0} + \sum_{n=1}^{\infty} = \sum_{n=\pm 1}^{\infty} \). Since
\[
\sum_{n=-\infty}^{\infty} \left( \frac{z}{n} \right)^2 < \infty,
\]
we may take \( \rho_n = 1 = \infty \)

\[
\sin \pi z = z e^{g(z)} \prod_{n=-\infty}^{\infty} E_n \left( \frac{z}{2n} \right) = z e^{g(z)} \prod_{n=-\infty}^{\infty} \left( 1 - \frac{2n}{z} \right)^{2n} \]

\[
= z e^{g(z)} \prod_{n=1}^{\infty} \left( 1 - \frac{2n}{z^2} \right).
\]

Thus \( \pi \sin \pi z = \pi z \prod \left( 1 - \frac{2n}{z^2} \right) \)
In fact, $q(z) = \log \prod (\text{see } \S C)$ \Rightarrow $\sum \pi_i z = \prod \pi_i z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$. 